marches aleatoires en milieu aleatoire et marches branchantes

Elie Aidekon

Thèse de doctorat de l'Université Paris VI

Spécialité :

Mathématiques

Présentée par

Elie Aïdékon

Pour obtenir le grade de

Docteur de l'Université Paris VI

Sujet de la thèse :

Marches aléatoires en milieu aléatoire et marches branchantes

---

Rapporteurs :	Mme	Nina	**GANTERT**	Universität Münster
	M.	Christophe	**SABOT**	Université Lyon I

soutenue publiquement le 27 mai 2009 devant le jury composé de

M. Francis	**COMETS**	Université Paris VII	Examinateur
M. Thomas	**DUQUESNE**	Université Paris VI	Examinateur
Mme Nina	**GANTERT**	Universität Münster	Rapporteur
M. Yueyun	**HU**	Université Paris XIII	Examinateur
M. Christophe	**SABOT**	Université Lyon I	Rapporteur
M. Zhan	**SHI**	Université Paris VI	Directeur

Remerciements

Je tiens à remercier particulièrement mon directeur de thèse, Zhan Shi, qui a eu la gentillesse de me guider tout au long de cette thèse. J'ai pu apprécier sa disponibilité, son écoute et sa rigueur, ce fut un réel plaisir de découvrir la recherche à ses côtés.

Je souhaite remercier Nina Gantert et Christophe Sabot d'avoir accepté de rapporter cette thèse. J'ai été très touché de l'attention qu'ils ont pu porter à mes travaux.

Je remercie tous les membres du jury, Francis Comets, Thomas Duquesne, Nina Gantert, Yueyun Hu, Christophe Sabot et Zhan Shi de me faire l'honneur de leur présence.

Je remercie le laboratoire de Probabilités et Modèles Aléatoires pour son accueil, en particulier Assane, Thanh Mai, Sophie, le bureau 4D1 passé et présent. Egalement tous les membres de l'administration qui m'ont été d'une aide précieuse.

Je remercie ceux qui m'ont accompagné durant ces années, spécialement Jean-Baptiste, Christophe, l'équipe de foot de l'ENS, les amis de NDM.

Le dernier remerciement va à ma famille dont le soutien, l'amour et la confiance ont énormément compté dans la conduite de ce projet.

Table des matières

Chapitre I

Introduction

Cette thèse porte sur deux modèles de marches aléatoires.

Notre premier modèle appartient à la famille des marches aléatoires en environnement aléatoire. Nous nous plaçons dans la situation où le graphe sur lequel évolue la marche est un arbre régulier ou de Galton–Watson, et nous intéressons aux propriétés asymptotiques de cette marche.

- Dans le cas transient, nous étudions la vitesse de la marche aléatoire. Nous obtenons un critère explicite pour avoir une vitesse non nulle, et donnons l'ordre de grandeur de la distance à la racine dans le régime à vitesse nulle. Nous appliquons nos résultats aux marches renforcées sur un arbre.

- Nous traitons ensuite des probabilités de grandes déviations de la marche. Nous évaluons le coût d'avoir une situation atypique de ralentissement ou d'accélération. Sous la probabilité annealed, nous distinguons les différents régimes de grandes déviations, typiques du modèle en environnement aléatoire.

La deuxième partie de ce travail présente un modèle de marches aléatoires branchantes avec absorption, réalisé en collaboration avec Bruno Jaffuel. Nous modélisons l'évolution d'une population se déplaçant sur l'axe des réels positifs, et dont les membres meurent lorsqu'ils passent l'origine. Deux régimes existent suivant la survie ou non de la population. En cas d'extinction totale de la population, nous cherchons à connaître les équivalents asymptotiques des probabilités de survie.

1 Marches aléatoires en milieu aléatoire sur un arbre

Les marches aléatoires en milieu aléatoire (MAMA) modélisent le mouvement d'une particule confrontée à un environnement inhomogène, mais dont les disparités d'ordre microscopique s'effacent à grande échelle. C'est un modèle qui attire par la diversité des régimes rencontrés, et qui contraste fortement avec les propriétés des marches aléatoires simples. La présence d'hétérogénéités dans le mileu favorisent l'apparition de pièges qui capturent la particule et entraînent de forts ralentissements de la marche.

Une MAMA se construit en deux temps. Dans un premier temps, on tire un environnement selon une certaine loi de probabilité. Cette loi est supposée décrire les propriétés statistiques du milieu. On fait ensuite partir notre particule selon une marche aléatoire dont les probabilités de transition sont données par l'environnement.

Ce modèle fut introduit originellement par le biologiste Chernov [Che67] afin de modéliser la réplication de l'ADN. En métallurgie, Temkin [Tem72] utilise le modèle pour étudier les transitions de phase dans les alliages. Citons également Le Doussal, Monthus et Fisher [LDMF99] pour des applications en physique.

Le modèle des MAMAs a été intensivement étudié en dimension 1, dont nous donnons un bref aperçu dans la première partie. Nous présenterons ensuite le cadre de notre travail, à savoir les MAMAs sur des arbres.

1.1 Rappel historique

On se place sur $\mathbb{Z} := \{\ldots, -2, -1, 0, +1, +2, \ldots\}$. A chaque site i, on tire de façon indépendante et identiquement distribuée (i.i.d.) un nombre $\omega(i)$ appartenant à $(0, 1)$. Ainsi, $\omega := (\omega(i), i \in \mathbb{Z})$ forme une famille de variables i.i.d. à valeurs dans $(0, 1)$, appelée l'environnement, et on note P sa distribution. Pour chaque réalisation de l'environnement ω, on définit la marche aléatoire $(X_n,\, n \geq 0)$ comme la chaîne de Markov vérifiant

$$P_\omega(X_{n+1} = i + 1 \,|\, X_n = i) = \omega(i) \;=\; 1 - P_\omega(X_{n+1} = i - 1 \,|\, X_n = i)\,.$$

La probabilité P_ω est appelée probabilité *quenched*. La probabilité *annealed* $\mathbb{P} := \int_\omega P_\omega(\cdot) P(d\omega)$ s'obtient en moyennant sur tous les environnements. La marche $(X_n,\, n \geq 0)$ est appelée *marche aléatoire en milieu aléatoire.* Indiquons que la MAMA ici définie évolue dans un environnement i.i.d., mais certains auteurs ([Ali99], [CGZ00]) étudient plus généralement la MAMA lorsque l'environnement est stationnaire et ergodique.

Le cas dégénéré où $\omega(0)$ est une masse de Dirac donne une marche aléatoire simple. Hormis ce cas, le modèle sous la loi annealed devient non-markovien. L'observation des valeurs passées de la marche donne des informations sur l'environnement déjà visité, et donc sur le comportement futur de la marche. C'est une des grandes particularités de ce modèle, et une différence majeure avec les marches classiques.

Soit

$$\rho_i := \frac{1-\omega(i)}{\omega(i)} \, .$$

Les variables ρ_i, $i \in \mathbb{Z}$ sont des variables i.i.d de l'environnement. Intuitivement, ρ_0 mesure la tendance qu'a l'environnement à pousser la marche vers la gauche. Solomon [Sol75] trouve le critère de transience/récurrence de la MAMA, lorsque l'espérance $E[\ln \rho_0]$ existe.

La marche $(X_n)_{n\geq 0}$ est récurrente si et seulement si $E[\ln(\rho_0)] = 0$.

Dans le cas transient ($E[\ln \rho_0] \neq 0$), Solomon montre de plus que la marche admet une vitesse asymptotique. Cette vitesse a la propriété d'être déterministe - elle est indépendante de l'environnement choisi -, et d'être explicite.

$$\exists\, v \in [-1,1] \quad \text{telle que} \quad \frac{X_n}{n} \longrightarrow v, \qquad n \to \infty \tag{1.1}$$

avec

$$v = \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} & \text{si } E[\rho_0] < 1 \\ 0 & \text{si } (E[\rho_0^{-1}])^{-1} \leq 1 \leq E[\rho_0] \\ \frac{E[\rho_0^{-1}]-1}{E[\rho_0^{-1}]+1} & \text{si } (E[\rho_0^{-1}])^{-1} > 1 \, . \end{cases} \tag{1.2}$$

Contrairement aux marches simples, la MAMA en dimension 1 peut donc partir vers l'infini avec une vitesse nulle. Ce résultat met en évidence la tendance du milieu aléatoire à ralentir la marche. Dans le cas récurrent, ce ralentissement a été observé par Sinai [Sin82] qui montre que la marche se comporte en $\log^2(n)$. Nous renvoyons à Golosov [Gol86] et Kesten [Kes86] pour l'étude de la convergence en loi, Hu et Shi [HS98] pour l'étude du maximum passé de la marche. Golosov [Gol84], et Andreoletti [And06] se sont intéressés à la localisation de la marche. Shi [Shi98], Gantert et Shi [GS02] , Dembo, Gantert, Peres et Shi [DGPS07] retrouvent ce phénomène grâce à l'étude du maximum du temps local.

Dans le cas transient à vitesse nulle, Kesten, Kozlov, et Spitzer [KKS75] ont révélé que la marche suivait un régime polynomial. Plus précisément, notons κ la solution de l'équation $E[\rho_0^\kappa] = 1$. Lorsque κ appartient à $(0, 1)$, on a sous des conditions techniques adéquates,

$$\frac{X_n}{n^\kappa} \xrightarrow{\mathcal{L}} \mathcal{L}_\kappa$$

où $\mathcal{L}_\kappa$ est une loi stable d'indice κ. Enriquez, Sabot et Zindy [ESZ07] ont retrouvé ce résultat par l'étude du potentiel et caractérisé la loi limite.

L'étude des grandes déviations concerne le comportement asymptotique des probabilités de fort ralentissement ou au contraire d'accélération de la marche.

Greven et Den Hollander [GdH94] établissent un principe de grandes déviations sous la loi quenched P_ω, et prouvent que la fonction de taux est déterministe. Comets, Gantert et Zeitouni [CGZ00] étendent le principe de grandes déviations quenched au cas où l'environnement est ergodique. Dans le même article, ils obtiennent le principe de grandes déviations sous la loi annealed, ainsi qu'un principe de grandes déviations fonctionnelles. Intuitivement, le phénomène de grandes déviations annealed résulte de la combinaison de deux phénomènes. On force d'abord l'environnement à se comporter comme un environnement ergodique η, ce qui a un coût mesuré par une fonction d'entropie $h(\eta|P)$. Etant donné cet environnement, l'effort restant est fait par la particule elle-même, ce qui fait intervenir la fonction de taux quenched I_η^q dans l'environnement η. Enfin, les auteurs montrent que les fonctions annealed et quenched sont différentes dès que la fonction annealed est non nulle. Cette différence s'explique encore par la possibilité de modifier l'environnement sous la loi annealed.

Lorsque ess inf $\rho_0 < 1 <$ ess sup ρ_0, le coût d'un ralentissement est sous-exponentiel. Dembo, Peres et Zeitouni [DPZ96] ont montré que dans ce cas (et si κ est pris strictement supérieur à 1), les probabilités annealed sont de l'ordre de $n^{1-\kappa}$. Dans la situation quenched, Gantert et Zeitouni [GZ98] prouvent que les probabilités de ralentissement deviennent de l'ordre de $\exp(-n^{1-1/\kappa})$.

1.2 Marches aléatoires en milieu aléatoire sur un arbre

Le modèle

Soit un arbre $\mathbb{T}$, c'est-à-dire un graphe qui ne présente aucun cycle. On suppose l'arbre enraciné et on note e cette racine. Pour chaque sommet x de l'arbre, on note $\overleftarrow{x}$ le parent de x, i.e. le premier sommet sur le chemin de x à e, $\nu(x)$ le nombre de ses enfants et $(x_i, 1 \leq i \leq \nu(x))$ ses enfants. On définit alors la MAMA sur l'arbre $\mathbb{T}$ de la façon suivante.

A chaque sommet x et de façon indépendante, on tire un vecteur de probabilités de transition $(\omega(x, \overleftarrow{x}), \omega(x, x_1), \ldots, \omega(x, x_{\nu(x)}))$ de somme égale à 1. Pour x_i, de père x et à une distance supérieure à 2 de la racine, on note

$$A(x_i) := \frac{\omega(x, x_i)}{\omega(x, \overleftarrow{x})}$$

et on suppose que les variables aléatoires $A(x)$, $x \in \mathbb{T}$ sont identiquement distribuées. Soit A une variable aléatoire ayant la distribution commune. On supposera une condition d'ellipticité de sorte qu'il existe $\alpha > 0$ tel que presque sûrement,

$$(H2) \qquad\qquad \alpha < A < 1/\alpha\,.$$

Enfin, on choisit autour de la racine n'importe quelles probabilités de transition non triviales (différentes de 0). Une fois l'environnement ω fixé, on définit la MAMA $(X_n,\, n \geq 0)$ partant de x comme la chaine de Markov vérifiant $X_0 = x$ et

$$P^x_\omega(X_{n+1} = z \,|\, X_n = y) = \omega(y, z)\,.$$

Ce modèle peut être vu comme la limite de la MAMA sur $\mathbb{Z}^d$ quand la dimension tend vers l'infini. Il a l'avantage de posséder une mesure réversible, ce qui permet d'utiliser le lien avec les réseaux électriques. Nous renvoyons à Doyle et Snell [DS84] pour une introduction à cette théorie, et à Lyons et Peres [LP04] pour l'application de cette théorie aux arbres. Petritis [Pet05] exhibe enfin un lien entre MAMA sur un arbre et grammaires quantiques, s'appliquant ainsi en génomique.

Le comportement de la MAMA sur un arbre dépend d'un paramètre particulier appelé nombre de branchement. On note $|x|$ la distance de x à la racine. Si on appelle ensemble de coupe tout ensemble Π de sommets de $\mathbb{T}$ tel que tout chemin infini de $\mathbb{T}$ rencontre Π, alors le nombre de branchement est défini par

$$br(\mathbb{T}) := \sup\left\{\lambda > 0 \,:\, \inf_{\Pi} \sum_{x\in\Pi} \lambda^{-|x|} > 0\right\}$$

où l'infimum porte sur tous les ensembles de coupe Π. Ce nombre qui dépend de la géométrie de l'arbre est une mesure de la croissance de l'arbre. Il a été introduit dans le cas des marches simples par Lyons [Lyo92]. Pour le cas des MAMAs, Lyons et Pemantle [LP92] obtiennent le critère de transience/ récurrence suivant :

- si $\inf_{[0,1]} E[A^t] < \frac{1}{br(\mathbb{T})}$, la MAMA est récurrente presque sûrement.
- si $\inf_{[0,1]} E[A^t] > \frac{1}{br(\mathbb{T})}$, la MAMA est transiente presque sûrement.

Le cas critique a été traité par Pemantle et Peres [PP95], mais la frontière n'est pas encore entièrement déterminée. Un cas particulier est l'arbre de Galton–Watson. Ici, le nombre de branchement vérifie $br(\mathbb{T}) = m$, où m est le nombre moyen d'enfants. Lyons et Pemantle [LP92] démontrent que

La MAMA sur un arbre de Galton–Watson est récurrente si et seulement si $\inf_{[0,1]} E[A^t] \leq \frac{1}{m}$.

Récemment, Menshikov et Petritis [MP02] ont retrouvé ce résultat pour l'arbre régulier en exhibant un lien entre MAMA et chaos multiplicatif.

Comportement de la MAMA sur un arbre régulier

Hu et Shi [HS07b], [HS07a] ont étudié le comportement de la MAMA récurrente sur l'arbre régulier. Par opposition avec la dimension 1, les auteurs ont obtenu une grande variété de régimes. Introduisons les notations nécessaires à l'énoncé des résultats. On considère un arbre $\mathbb{T}$ b-régulier, c'est-à-dire tel que chaque sommet admet b enfants. Le maximum passé de la marche est donné par

$$X_n^* := \max_{0\leq k\leq n} |X_k|\,.$$

On note $\phi(t) := \ln(E[A^t])$ la transformée de Laplace de $\ln A$, et on définit

$$\mu := \inf\{t > 1 \,:\, E[A^t] = \frac{1}{b}\}$$

On observe trois grand régimes selon la forme de ϕ. Nous prenons la liberté de ne pas préciser davantage la nature des limites ci-dessous.

– Si $\inf_{[0,1]} E[A^t] < \frac{1}{b}$, X_n^* se comporte en $\ln n$.

– Si $\inf_{[0,1]} E[A^t] = \frac{1}{b}$ et $\phi'(1) < 0$, X_n^* se comporte en n^δ avec $\delta := 1 - \frac{1}{\min(\mu,2)}$.

– Si $\inf_{[0,1]} E[A^t] = \frac{1}{b}$ et $\phi'(1) \geq 0$, X_n^* se comporte en $\ln^3(n)$.

En particulier, le cas récurrent présente un régime à la Kesten-Kozlov-Spitzer similaire au régime transient de la dimension 1. Notons qu'aucune convergence en loi n'est encore connue dans le cadre des MAMAs récurrentes sur un arbre.

Dans le cas transient, T. Gross [Gro04] obtient la loi des grands nombres grâce à la construction de temps de renouvellement. Il montre que la marche admet une vitesse

$$v = \lim_{n\to\infty} \frac{|X_n|}{n}.$$

Un des objectifs de cette thèse est de savoir si cette vitesse est strictement positive.

La marche biaisée sur un arbre de Galton–Watson

La marche biaisée sur un arbre (non nécessairement de Galton–Watson) peut être vue comme un cas particulier des MAMAs sur un arbre, en prenant A déterministe. Historiquement, ce modèle a été étudié avant les MAMAs sur les arbres, et les critères de transience /récurrence ont été obtenus par Lyons [Lyo92]. Notons λ le biais vers la racine (ce qui revient à prendre $A = \lambda^{-1}$).

La marche biaisée est transiente si et seulement si $\lambda < br(\mathbb{T})$.

Lorsque $\mathbb{T}$ est un arbre de Galton–Watson, nous rappelons que $br(\mathbb{T})$ est simplement le nombre moyen d'enfants. Dans ce cas, Lyons, Pemantle et Peres [LPP95], [LPP96] ont étudié le régime transient. Notons $q := (q_k, k \geq 0)$ la distribution du nombre d'enfants, $f(s) := \sum_{k\geq 0} q_k s^k$ la fonction génératrice et p la probabilité d'extinction de l'arbre. Deux situations sont à considérer.

– Si $q_0 = 0$, la marche biaisée transiente sur un arbre de Galton–Watson a nécessairement une vitesse positive.

– Si $q_0 > 0$, la marche biaisée transiente a une vitesse non nulle si et seulement si $f'(p) < \lambda$.

On remarque donc que la vitesse est toujours positive lorsque l'arbre n'a pas de feuilles. Dans le cas contraire, et lorsque $\lambda < 1$, la marche peut être ralentie dans les parties finies de l'arbre, ce qui explique les situations à vitesse nulle. Dans le cas de la marche simple ($\lambda = 1$) et si $q_0 = 0$, cette vitesse est connue explicitement et donnée par

$$v = \sum_{k\geq 1} q_k \frac{k-1}{k+1}\,.$$

Dans le cas général, Chen [Che97] puis Virag [Vir00] donnent une borne supérieure pour la vitesse. Notons v la vitesse, et plaçons-nous dans le cas sans feuilles $q_0 = 0$. Alors

$$v \leq \frac{m-\lambda}{m+\lambda}\,. \tag{1.3}$$

Mentionnons que la majoration est un cas particulier de l'article de Virag, qui donne une majoration sur un graphe général en termes de nombre essentiel de branchement. La majoration indique que la marche biaisée sur un arbre de Galton–Watson est plus lente que la marche biaisée sur un arbre 'régulier' ayant le même nombre de branchement. C'est encore une illustration du ralentissement du milieu aléatoire. Toujours dans le cas transient, Piau [Pia98] établit un théorème central limite pour la marche simple, grâce à l'étude de la queue de distribution des temps de renouvellement. Peres et Zeitouni [PZ08] obtiennent le théorème central limite dans le cas transient général, ainsi que dans le cas nul récurrent grâce à la construction d'une probabilité invariante pour l'environnement vu de la particule. Dans les cas d'arbres de Galton–Watson avec feuilles, Ben Arous, Fribergh, Gantert et Hammond [BAFGH08] étudient la convergence en loi de la marche transiente.

Dembo, Gantert, Peres et Zeitouni [DGPZ02] établissent également un principe de grandes déviations. Il est intéressant de noter que contrairement à la dimension 1, les auteurs montrent que les fonctions de taux quenched et annealed sont identiques en général. Cette égalité provient d'un phénomène d'incertitude sur la localisation du point d'atteinte du niveau n. Nous verrons que ce phénomène n'a plus nécessairement lieu pour les MAMAs sur un arbre. Enfin, en se restreignant à un rayon de l'arbre, Dembo, Gantert et Zeitouni [DGZ04] retrouvent des fonctions de taux quenched et annealed différentes.

1.3 Présentation des résultats en milieu aléatoire

Chapitre II : Vitesse de la MAMA sur un arbre de Galton–Watson

Nous nous sommes d'abord intéressés au comportement de la MAMA transiente sur un arbre b-régulier. Nous supposerons dans toute la suite que la condition $(H2)$ est toujours vérifiée. Contrairement à la dimension 1, nous montrons que la vitesse est toujours positive.

Théorème 1. *La vitesse de la MAMA transiente sur un arbre b-régulier est strictement positive dès que* $b \geq 2$.

La croissance exponentielle de l'arbre régulier empêche la création de trappes pour la particule. Nous avons voulu savoir si le résultat était encore vrai pour un arbre général de Galton–Watson sans feuilles ($q_0 = 0$). Si $q_1 > 0$, des pièges apparaissent, constituées par des longs tubes où chaque sommet a un seul enfant. On montre que ces pièges peuvent ralentir assez la marche pour avoir une vitesse nulle, et donnons le critère adéquat. On définit

$$\Lambda := Leb \left\{ t \in \mathbb{R} : E[A^t] \leq \frac{1}{q_1} \right\}. \tag{1.4}$$

Si $q_1 = 0$, on pose $\Lambda := \infty$.

Théorème 2. *Supposons que* $\inf_{[0,1]} E[A^t] > \frac{1}{m}$, $q_0 = 0$, *et soit* Λ *défini par* (1.4).

(a) *Si* $\Lambda \leq 1$, *la MAMA sur l'arbre de Galton–Watson a une vitesse nulle.*

(b) *Si* $\Lambda > 1$, *la marche admet une vitesse strictement positive.*

La MAMA fournit donc un exemple de marche transiente à vitesse nulle sur un graphe à croissance exponentielle. Expliquons d'où vient ce critère. La MAMA restreinte à un piège devient une MAMA uni-dimensionnelle. De plus, la longueur du piège est une variable géométrique de paramètre q_1. On est donc ramené au calcul d'un temps de sortie d'une MAMA en dimension 1 de l'intervalle $[0, G]$ où G est une variable géométrique. Lorsque $\Lambda \leq 1$, on montre que ce temps de sortie est de moyenne annealed infinie. Une application de la loi des grands nombres montre alors que la vitesse est nulle. Lorsque $\Lambda > 1$, la particule s'échappe en un temps fini des pièges. Les pièges ne retenant plus la particule, on est alors plus ou moins ramené à l'étude d'une MAMA sur l'arbre obtenu en enlevant les tubes. Sur cet arbre, tous les sommets ont au moins deux enfants, et la MAMA sur un tel arbre admet une vitesse strictement positive.

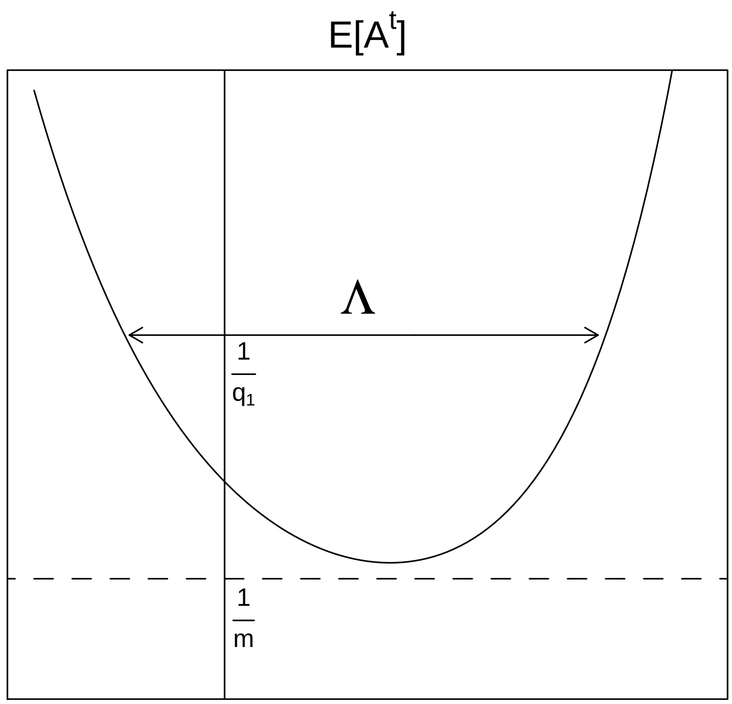

Figure I.1 – L'exposant Λ

Nous n'avons pas d'expression explicite pour la vitesse. Nous obtenons cependant dans le chapitre III une inégalité faisant intervenir la conductance de l'arbre. Elle s'écrit :

Proposition 1.1. *Soit v la vitesse de la MAMA sur un arbre de Galton–Watson, et β la conductance de l'arbre, i.e. $\beta(x) = P_\omega^x(T_{\overleftarrow{x}} = \infty)$. Alors,*

$$\frac{2}{E\left[\beta\right]} - 1 \leq \frac{1}{v} \leq E\left[\frac{2}{\beta}\right] - 1\,.$$

Grâce à l'étude de la conductance, on montre

Corollaire 1.2. *La vitesse vérifie*

$$\frac{E\left[\sum_{i=1}^{\nu(x)} A(x_i)\right] - 1}{E\left[\sum_{i=1}^{\nu(x)} A(x_i)\right] + 1} \geq v \geq \frac{1 - E\left[\frac{1}{\sum_{i=1}^{\nu(x)} A(x_i)}\right]}{1 + E\left[\frac{1}{\sum_{i=1}^{\nu(x)} A(x_i)}\right]}$$

(L'environnement étant stationnaire, l'encadrement ne dépend pas de x).

Dans le cas des marches biaisées, celle-ci permet de retrouver l'équation (1.3). La proposition 1.1 et son corollaire seront démontrés dans le Chapitre III.

Dans le cas à vitesse nulle, nous observons un phénomène à la Kesten-Kozlov-Spitzer, où la marche a une croissance polynômiale.

Théorème 3. *Si* $\inf_{[0,1]} E[A^t] > \frac{1}{m}$, *et* $\Lambda \leq 1$, *alors*

$$\lim_{n\to\infty} \frac{\ln(|X_n|)}{\ln(n)} = \Lambda \quad p.s.$$

Remarquons le parallèle avec la MAMA en dimension 1, le paramètre Λ faisant écho au κ de Kesten-Kozlov-Spitzer.

Nous nous sommes enfin intéressés à la vitesse de la marche linéaire renforcée sur l'arbre régulier. La marche renforcée est un modèle de marche aléatoire introduit par Coppersmith et Diaconis [CD87], où la particule est favorisée à passer par des sites déjà visités. On se place sur l'arbre b-régulier. Chaque arête est initialement chargée d'un poids égal à 1. Lorsque la particule traverse une arête, celle-ci voit son poids augmenter d'une constante $\delta > 0$. Etant donnés les poids à un instant donné, la particule au site x choisit de se déplacer au site y avec une probabilité proportionnelle au poids de l'arête (x, y). Ce processus est appelée marche linéaire renforcée. Pemantle [Pem88] étudie la transience/récurrence de la marche et met en évidence un paramètre critique δ_c au-delà duquel la marche est récurrente. Dans le cas $\delta = 1$, pour lequel la marche est presque sûrement transiente sur tout arbre régulier, $b \geq 2$, Collevecchio [Col06] a montré que la vitesse était strictement positive dès que $b \geq 70$. Nous montrons que le résultat est vrai pour tout $b \geq 2$.

Théorème 4. *La marche aléatoire linéairement renforcée sur l'arbre régulier admet une vitesse strictement positive dès que* $b \geq 2$.

Ce théorème repose sur le lien entre marche renforcée et MAMA exhibé par Pemantle, grâce aux urnes de Polya. Pemantle montre qu'une marche renforcée a la loi de la MAMA telle que pour $y \neq e$, la densité de $\omega(y, z)$ sur $(0, 1)$ est donnée par

- $f_0(x) = \frac{b}{2}\,(1-x)^{\frac{b}{2}-1}$ si $z = \overleftarrow{y}$,
- $f_1(x) = \frac{\Gamma(\frac{b}{2}+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{b+1}{2})}\, x^{-\frac{1}{2}}(1-x)^{\frac{b-1}{2}}$ si z est un enfant de y.

Chapitre III : Etude des grandes déviations

Notre deuxième travail a été d'étudier les situations de ralentissement ou d'accélération de la marche. Nous obtenons un principe de grandes déviations annealed et quenched pour les temps d'atteinte. Nous considérons un arbre de Galton–Watson sans feuilles. Nous distinguons les cas d'accélération et de ralentissement. Rappelons aussi que v désigne la vitesse de la MAMA et est déterministe.

Théorème 5. (Accélération de la marche) *Il existe deux fonctions continues, convexes et strictement décroissantes $I_a \leq I_q$ de $[1, 1/v]$ dans $\mathbb{R}_+$, telles que $I_a(1/v) = I_q(1/v) = 0$ et pour tout $a < b$, $b \in [1, 1/v]$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}^e\left(\frac{\tau_n}{n} \in]a, b]\right)\right) = -I_a(b)\,, \tag{1.5}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(\frac{\tau_n}{n} \in]a, b]\right)\right) = -I_q(b)\,. \tag{1.6}$$

En particulier, une accélération de la marche a toujours un coût exponentiel. Ceci n'est plus vrai pour un ralentissement. Soit ν_{min} le plus petit entier k tel que $q_k > 0$ ($\nu_{min} \geq 1$).

Théorème 6. (Ralentissement de la marche) *Il existe deux fonctions continues, convexes $I_a \leq I_q$ de $[1/v, +\infty[$ dans $\mathbb{R}_+$, telles que $I_a(1/v) = I_q(1/v) = 0$ et pour tous $1/v \leq a < b$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}^e\left(\frac{\tau_n}{n} \in [a, b[\right)\right) = -I_a(a)\,, \tag{1.7}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(\frac{\tau_n}{n} \in [a, b[\right)\right) = -I_q(a)\,. \tag{1.8}$$

Si ess inf $A > {\nu_{min}}^{-1}$, les deux fonctions sont strictement croissantes. Sinon, elles sont égales à zéro.

En rapport avec le travail de Dembo et al. [DGPZ02], nous aimerions savoir si les deux fonctions de taux sont identiques. La situation au point $a = 1$ montre que ce n'est pas toujours le cas. Soit

$$\psi(\theta) := \ln\left(E_P\left[\sum_{i=1}^{\nu(e)} \omega(e, e_i)^\theta\right]\right).$$

On a donc $\psi(0) = \ln(m)$ et $\psi(1) = \ln\left(E_P\left[\sum_{i=1}^{\nu(e)} \omega(e, e_i)\right]\right)$.

Proposition 1.3. *Les fonctions de taux au point* 1 *sont égales à*

$$I_a(1) = -\psi(1)\,, \tag{1.9}$$

$$I_q(1) = -\inf_{]0,1]} \frac{1}{\theta}\psi(\theta)\,. \tag{1.10}$$

En particulier, $I_a(1) = I_q(1)$ *si et seulement si* $\psi'(1) \leq \psi(1)$*. Dans le cas contraire, on a* $I_a(1) < I_q(1)$*.*

De façon plutôt surprenante, on montre ainsi que même sur un arbre régulier, les fonctions de taux peuvent être différentes. Ceci indique qu'il n'existe pas de phénomène d'incertitude sur le point d'atteinte du niveau n comme dans [DGPZ02]. On assisterait donc à un phénomène de localisation sur l'arbre régulier, conditionnellement à une accélération. Cependant, nous ne connaissons pas de critère pour caractériser l'égalité des fonctions de taux.

D'après le théorème 6, le ralentissement de la marche peut avoir un coût sous-exponentiel. Nous cherchons alors l'ordre de grandeur des probabilités annealed. Pour éviter la structure locale autour de la racine, nous nous plaçons sur l'évènement E où la marche ne revient plus à la racine.

Théorème 7. *Supposons que ess inf* $A < \nu_{min}^{-1}$*.*

(i) Si on est dans l'un des cas "ess inf $A < \nu_{min}^{-1}$ *et* $q_1 = 0$*" ou "ess inf* $A < \nu_{min}^{-1}$ *et ess sup* $A < 1$*", alors il existe deux constantes* $c_1, c_2 \in (0,1)$ *telles que pour tout* $a > 1/v$ *et* n *assez grand,*

$$e^{-n^{c_1}} < \mathbb{P}^e(\tau_n > an \,|\, E) < e^{-n^{c_2}}\,. \tag{1.11}$$

(ii) Si $q_1 > 0$ *et ess sup* $A > 1$ *(id est* $\Lambda < \infty$*), alors le régime est polynômial et on a pour tout* $a > 1/v$*,*

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \ln\left(\mathbb{P}^e(\tau_n > an \,|\, E)\right) = 1 - \Lambda\,. \tag{1.12}$$

Nous remarquons que nous obtenons le régime polynômial de la dimension 1 lorsque Λ est fini. Dans le cas (i), nous n'avons pas d'estimation plus précise des constantes c_1 et c_2.

Enfin, le chapitre III termine avec la preuve de la Proposition 1.1 déjà citée.

2 Marches branchantes avec absorption

La deuxième partie de ce travail traite des marches branchantes avec absorption appelées aussi marches branchantes avec barrière. Le modèle en temps discret a été introduit par Biggins, Lubachevsky, Schwartz et Weiss [BLSW91], pour des problèmes de simulation informatique. Présentons le modèle.

À $n = 0$, il existe une particule, l'ancêtre, située en $x > 0$. Celle-ci donne naissance à b enfants, où b est une constante, puis disparaît. Au temps $n = 1$, les enfants se déplacent de façon i.i.d. Les particules qui atterrissent sur un réel négatif meurent instantanément, sans donner de descendance. Puis le processus se répète.

Mathématiquement, soit $\mathbb{T}$ un arbre b-régulier (représentant la descendance de l'ancêtre), et $(X_u,\, u \in \mathbb{T}\backslash\{e\})$ une famille de variables aléatoires i.i.d. Pour tout u, soit $u_0 = e, u_1, \ldots, u_{|u|} = u$ le chemin de e à u. On définit $I(u) := \inf_{k=1\ldots|u|}\{x + X_{u_1} + \ldots + X_{u_k}\}$ et

$$S(u) = \mathbb{1}_{\{I(u)>0\}}(x + X_{u_1} + \ldots + X_u)\,.$$

La variable $S(u)$ représente la position de la particule u. Toutes les particules dont la position est 0 sont considérées comme mortes. La première question est de savoir si la population a une chance de survivre indéfiniment. Pour cela, on suppose que le déplacement X est tel que $\sup\{\theta \geq 0 \,:\, E[e^{\theta X}] < \infty\} =: \theta_c > 0$ et on pose

$$\gamma := \inf_{\theta \geq 0} E[e^{\theta X}]\,.$$

Biggins et al [BLSW91] montrent que

- Si $\gamma < 1/b$, alors la population meurt presque sûrement (*cas sous-critique*).
- Si $\gamma > 1/b$, alors la population survit avec probabilité strictement positive (*cas sur-critique*).

Le cas critique correspond à $\gamma = 1/b$, et implique que X est de moyenne négative. Lorsque l'on suppose en plus (ce que l'on fera) que la distribution de X est non-arithmétique, et que l'infimum est atteint en un point $0 < \nu < \theta_c$, alors il y a également extinction presque sûre de la population.

Dans les cas critique et sous-critique, nous avons étudié les probabilités d'avoir un survivant au bout d'un temps long. Ces questions ont été traitées pour l'analogue continu de notre modèle, à savoir la diffusion branchante en présence d'une barrière. Si on considère un mouvement brownien de drift $-\rho$, et subissant un branchement dyadique de taux $\beta > 0$, le cas sous-critique correspondant à $\rho > \sqrt{2\beta}$, a été étudié par Harris et Harris [HH07]. Les auteurs ont montré que la probabilité de survie au temps t est exponentiellement petite. Plus précisément, en notant Z_t la population au temps t, il existe une constante $K_1 > 0$ indépendante de x telle que

$$P^x(Z_t > 0)\frac{\sqrt{2\pi t^3}}{x}e^{-\rho x+(\frac{1}{2}\rho^2-\beta)t} \to K_1\,.$$

Cette convergence peut se réécrire $E[Z_t \,|\, Z_t > 0] \to K_2$, pour une constante $K_2 > 0$. Conditionnellement à la survie, la population moyenne tend vers une constante. Cette caractéristique n'est plus vraie dans le cas critique. En effet, Kesten [Kes78] montre qu'il existe une constante $C > 0$ telle que

$$xe^{\rho x - C\ln(t)^2} \leq P(Z_t > 0)\exp(\frac{3\rho^2\pi^2}{2})^{1/3}t^{1/3} \leq (1+x)e^{\rho x + C\ln(t)^2}\,.$$

Cette estimée contraste avec l'espérance $E[Z_t]$ qui est de l'ordre de $t^{-3/2}$.

Les diffusions branchantes en présence d'une barrière ont également intéressé certains physiciens à l'image de Derrida et Simon[DS07], [SD08] grâce au lien qui existe avec l'équation F-KPP. De plus, par changement de probabilité, il est possible de voir ces marches par construction d'épine. Pour des exemples de telles méthodes, on se réfèrera par exemple à Kurtz et al [KLPP97], Hu et Shi [HS] et Biggins et Kyprianou [BK04]. Dans le cadre du mouvement brownien branchant sans barrière, Chauvin et Rouault [CR88] ont étudié les probabilités de grandes déviations sur la vitesse de la particule la plus à droite.

Chapitre IV : Probabilités de survie des marches branchantes avec barrière

Nous revenons désormais au cas discret. Nous avons montré que l'ordre de grandeur des probabilités de survie était le même que dans le cas continu. Soit X_i, $i \geq 0$ une suite de variables i.i.d de distribution la loi de déplacement d'une particule du processus de branchement. On définit, en posant $I_k := \min_{0\leq i\leq k} X_i$,

$$\widetilde{V}(x) := 1 + \sum_{k=0}^{\infty}\frac{1}{\gamma^k}E\left[e^{\nu I_k}\,, S_k = I_k \geq -x\right]\,.$$

Théorème 8 (Cas sous-critique)**.** *Si $\gamma < 1/b$, alors il existe une constante $C_1 > 0$ indépendante de x telle que*

$$P^x(Z_n > 0) \sim C_1 e^{\nu x}\widetilde{V}(x)b^n\gamma^n n^{-3/2}\,.$$

pour tout $x > 0$ *point de continuité de la fonction* $\widetilde{V}$.

Théorème 9 (Cas critique)**.** *Si* $\gamma = 1/b$, *alors pour tout* $x > 0$,

$$\ln(P^x(Z_n > 0)) \sim -\left(\frac{3\nu^2\pi^2\sigma^2}{2}\right)^{1/3} n^{1/3}$$

avec $\sigma^2 := E\left[X^2 e^{\nu X}\right]/\gamma$.

Un outil prépondérant dans ce travail a été l'étude des marches avec drift négatif conditionnées à rester positives par Iglehart [Igl74].

Première partie

Marches aléatoires en milieu aléatoire sur un arbre

Chapitre II

Transient random walks in random environment on a Galton–Watson tree [1]

Summary. We consider a transient random walk (X_n) in random environment on a Galton–Watson tree. Under fairly general assumptions, we give a sharp and explicit criterion for the asymptotic speed to be positive. As a consequence, situations with zero speed are revealed to occur. In such cases, we prove that X_n is of order of magnitude n^{Λ}, with $\Lambda \in (0, 1)$. We also show that the linearly edge reinforced random walk on a regular tree always has a positive asymptotic speed, which improves a recent result of Collevecchio [Col06].

Key words. Random walk in random environment, reinforced random walk, law of large numbers, Galton–Watson tree.

AMS subject classifications. 60K37, 60J80, 60F15.

1 Introduction

1.1 Random walk in random environment

Let ν be an $\mathbb{N}^*$-valued random variable (with $\mathbb{N}^* := \{1, 2, \cdots\}$) and $(A_i, i \geq 1)$ be a random variable taking values in $\mathbb{R}_+^{\mathbb{N}^*}$. Let $q_k := P(\nu = k)$, $k \in \mathbb{N}^*$. We assume $q_0 = 0$, $q_1 < 1$, and $m := \sum_{k \geq 0} k q_k < \infty$. Writing $V := (A_i, i \leq \nu)$, we construct a Galton–Watson tree as follows.

1. The bulk of this chapter is to appear in *Probability Theory and Related Fields* ([Aid08]). We append Section 9 to deal with the critical case left open in the original work.

Let e be a point called the root. We pick a random variable $V(e) := (A(e_i), i \leq \nu(e))$ distributed as V, and draw $\nu(e)$ children to e. To each child e_i of e, we attach the random variable $A(e_i)$. Suppose that we are at the n-th generation. For each vertex x of the n-th generation, we pick independently a random vector $V(x) = (A(x_i), i \leq \nu(x))$ distributed as V, associate $\nu(x)$ children $(x_i, i \leq \nu(x))$ to x, and attach the random variable $A(x_i)$ to the child x_i. This leads to a Galton–Watson tree $\mathbb{T}$ of offspring distribution q, on which each vertex $x \neq e$ is marked with a random variable $A(x)$.

We denote by GW the distribution of $\mathbb{T}$. For any vertex $x \in \mathbb{T}$, let $\overleftarrow{x}$ be the parent of x and $|x|$ its generation ($|e| = 0$). In order to make the presentation easier, we artificially add a parent $\overleftarrow{e}$ to the root e. We define the environment ω by $\omega(\overleftarrow{e}, e) = 1$ and for any vertex $x \in \mathbb{T} \backslash \{\overleftarrow{e}\}$,

– $\omega(x, x_i) = \frac{A(x_i)}{1+\sum_{i=1}^{\nu(x)} A(x_i)}$, $\forall\, 1 \leq i \leq \nu(x)$,

– $\omega(x, \overleftarrow{x}) = \frac{1}{1+\sum_{i=1}^{b} A(x_i)}$.

For any vertex $y \in \mathbb{T}$, we define on $\mathbb{T}$ the Markov chain $(X_n,\, n \geq 0)$ starting from y by

$$\begin{aligned} P_\omega^y(X_0 = y) &= 1, \\ P_\omega^y(X_{n+1} = z \,|\, X_n = x) &= \omega(x, z)\,. \end{aligned}$$

Given $\mathbb{T}$, $(X_n,\, n \geq 0)$ is a $\mathbb{T}$-valued random walk in random environment (RWRE). We note from the construction that $\omega(x, .)$, $x \neq \overleftarrow{e}$ are independent.

Following [LP92], we also suppose that $A(x)$, $x \in \mathbb{T}$, $|x| \geq 1$, are identically distributed. Let A denote a random variable having the common distribution. We assume the existence of $\alpha > 0$ such that ess sup$(A) \leq \alpha$ and ess sup$(\frac{1}{A}) \leq \alpha$. The following criterion is known.

Theorem A (Lyons and Pemantle [LP92]) *The walk (X_n) is transient if $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, and is recurrent otherwise.*

When $\mathbb{T}$ is a regular tree, Menshikov and Petritis [MP02] obtain the transience/recurrence criterion by means of a relationship between the RWRE and Mandelbrot's multiplicative cascades ; Hu and Shi [HS07a],[HS07b] characterize different asymptotics of the walk in the recurrent case, revealing a wide range of regimes.

Throughout the paper, we assume that the walk is transient (i.e., $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$ according to Theorem A). Given the transience, natural questions arise concerning the rate of escape of the walk. The law of large numbers says that there exists a deterministic $v \geq 0$

(which can be zero) such that

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v, \qquad a.s.$$

This was proved by Gross [Gro04] when $\mathbb{T}$ is a regular tree, and by Lyons et al. [LPP96] when A is deterministic ; their arguments can be easily extended in the general case (i.e., when $\mathbb{T}$ is a Galton–Watson tree and A is random).

We are interested in determining whether $v > 0$.

When A is deterministic, it is shown by Lyons et al. [LPP96] that the transient random walk always has positive speed. Later, an interesting large deviation principle is obtained in Dembo et al. [DGPZ02]. In the special case of non-biased random walk, Lyons et al. [LPP95] succeed in computing the value of the speed.

We recall two results for RWRE on $\mathbb{Z}$ (which can be seen as a half line-tree). The first one gives a necessary and sufficient condition for RWRE to have positive asymptotic speed.

Theorem B (Solomon [Sol75]) *If* $\mathbb{T} = \mathbb{Z}$*, then*

$$\mathbf{E}\left[\frac{1}{A}\right] < 1 \Longleftrightarrow \lim_{n\to\infty} \frac{X_n}{n} > 0 \quad a.s.$$

When the transient RWRE has zero speed, Kesten, Kozlov and Spitzer in [KKS75] prove that the walk is of polynomial order. To this end, let $\kappa \in (0,1]$ be such that $E\left[\frac{1}{A^\kappa}\right] = 1$. Under some mild conditions on A,

– if $\kappa < 1$, then $\frac{X_n}{n^\kappa}$ converges in distribution.
– If $\kappa = 1$, then $\frac{\ln(n) X_n}{n}$ converges in probability to a positive constant.

The aim of this paper is to study the behaviour of the transient random walk when $\mathbb{T}$ is a Galton–Watson tree. Let *Leb* represent the Lebesgue measure on $\mathbb{R}$ and let

$$\Lambda := Leb\left\{t \in \mathbb{R} : \mathbf{E}[A^t] \le \frac{1}{q_1}\right\}. \tag{1.1}$$

If $q_1 = 0$, then we define $\Lambda := \infty$. Notice that this definition is similar to the definition of κ in the one-dimensional setting. Our first result, which is a (slightly weaker) analogue of Solomon's criterion for Galton–Watson tree $\mathbb{T}$, is stated as follows.

Theorem 1.1. *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$*, and let* Λ *be as in* (1.1).

(a) *If* $\Lambda \le 1$*, the walk has zero speed.*
(b) *If* $\Lambda > 1$*, the walk has positive speed.*

Corollary 1.2. *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$. *If* $\mathbb{T}$ *is a regular tree, then the walk has positive speed.*

Theorem 1.1 extends Theorem B. Corollary 1.2 says there is no Kesten–Kozlov–Spitzer-type regime for RWRE when the tree is regular. Our next result exhibits such a regime for Galton–Watson trees $\mathbb{T}$.

Theorem 1.3. *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, *and* $\Lambda \leq 1$. *Then*

$$\lim_{n\to\infty} \frac{\ln(|X_n|)}{\ln(n)} = \Lambda \quad a.s.$$

Since $\Lambda > 0$, the walk is proved to be of polynomial order. As expected, Λ plays the same role as κ.

1.2 Linearly edge reinforced random walk

The reinforced random walk is a model of random walk introduced by Coppersmith and Diaconis [CD87] where the particle tends to jump to familiar vertices. We consider the case where the graph is a b-ary tree $\mathbb{T}$, that is a tree where each vertex has b children ($b \geq 2$). At each edge (x, y), we initially assign the weight $\pi(x, y) = 1$. If we know the weights and the position of the walk at time n, we choose an edge emanating from X_n with probability proportional to its weight. The weight of the edge crossed by the walk then increases by a constant $\delta > 0$. This process is called the Linearly Edge Reinforced Random Walk (LERRW). Pemantle in [Pem88] proves that there exists a real δ_0 such that the LERRW is transient if $\delta < \delta_0$ and recurrent if $\delta > \delta_0$ ($\delta_0 = 4,29..$ for the binary tree). We focus, from now on, on the case $\delta = 1$, so that the LERRW almost surely is transient. Recently, Collevecchio in [Col06] shows that when $b \geq 70$ the LERRW has a positive speed v which verifies $0 < v \leq \frac{b}{b+2}$. We propose to extend the positivity of the speed to any $b \geq 2$.

Theorem 1.4. *The linearly edge reinforced random walk on a b-ary tree has positive speed.*

We rely on a correspondence between RWRE and LERRW, explained in [Pem88]. By means of a Polya's urn model, Pemantle shows that the LERRW has the distribution of a certain RWRE, such that for any $y \neq \overleftarrow{e}$, the density of $\omega(y, z)$ on $(0, 1)$ is given by

– $f_0(x) = \frac{b}{2}(1-x)^{\frac{b}{2}-1}$ if $z = \overleftarrow{y}$,

– $f_1(x) = \frac{\Gamma(\frac{b}{2}+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{b+1}{2})} x^{-\frac{1}{2}}(1-x)^{\frac{b-1}{2}}$ if z is a child of y.

Consequently, we only have to prove the positivity of the speed of this RWRE.

With the notation of Section 1.1, A is not bounded in this case, which means Theorem 1.1 does not apply. To overcome this difficulty, we prove the following result.

Theorem 1.5. *Let $\mathbb{T}$ be a b-ary tree and assume that $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{b}$ and*

$$E\left[\left(\sum_{i=1}^{b} A_i\right)^{-1}\right] < \infty\,.$$

Then the RWRE has positive speed.

Since the RWRE associated with the LERRW satisfies the assumptions of Theorem 1.5 as soon as $b \geq 3$, Theorem 1.4 follows immediately in the case $b \geq 3$. The case of the binary tree is dealt with separately.

The rest of the paper is organized as follows. We prove Theorem 1.5 in Section 2. In Section 3, we prove the upper bound in Theorem 1.3. Some technical results are presented in Section 4, and are useful in Section 5 in the proof of the lower bound in Theorem 1.3. In Section 6, we prove Theorem 1.1 except for the critical case $\Lambda = 1$. The proof of Theorem 1.4 for the binary tree is the subject of Section 7. Section 8 is devoted to the computation of parameters used in the proof of Theorem 1.3. Finally, Section 9 deals with the critical case left open in Section 6.

2 The regular case, and the proof of Theorem 1.5

We begin the section by giving some notation. Let $\mathbf{P}$ denote the distribution of ω conditionally on $\mathbb{T}$, and $\mathbb{P}^x$ the law defined by $\mathbb{P}^x(\cdot) := \int P^x_\omega(\cdot)\mathbf{P}(\mathrm{d}\omega)$. We emphasize that P^x_ω, $\mathbf{P}$ and $\mathbb{P}^x$ depend on $\mathbb{T}$. We respectively associate the expectations E^x_ω, $\mathbf{E}$, $\mathbb{E}^x$. We denote also by $\mathbf{Q}$ and $\mathbb{Q}^x$ the measures :

$$\begin{aligned} \mathbf{Q}(\cdot) &:= \int \mathbf{P}(\cdot) GW(\mathrm{d}\mathbb{T})\,, \\ \mathbb{Q}^x(\cdot) &:= \int \mathbb{P}^x(\cdot) GW(\mathrm{d}\mathbb{T})\,. \end{aligned}$$

For sake of brevity, we will write $\mathbb{P}$ and $\mathbb{Q}$ for $\mathbb{P}^e$ and $\mathbb{Q}^e$.

Define for $x, y \in \mathbb{T}$, and $n \geq 1$,

$$\begin{aligned} Z_n &:= \#\{x \in \mathbb{T} : \ |x| = n\}, \\ x \leq y &\Leftrightarrow \exists\, p \geq 0,\ \exists\, x = x_0, \ldots, x_p = y \in \mathbb{T} \text{ such that } \forall\, 0 \leq i < p,\ x_i = \overleftarrow{x}_{i+1}. \end{aligned}$$

If $x \leq y$, we denote by $[\![x, y]\!]$ the set $\{x_0, x_1, \ldots, x_p\}$, and say that $x < y$ if moreover $x \neq y$.

Define for $x \neq \overleftarrow{e}$, and $n \geq 1$,

$$\begin{aligned} T_x &:= \inf\{k \geq 0 : \ X_k = x\}, \\ T_x^* &:= \inf\{k \geq 1 : \ X_k = x\}, \\ \beta(x) &:= P_\omega^x(T_{\overleftarrow{x}} = \infty). \end{aligned}$$

We observe that $\beta(x)$, $x \in \mathbb{T} \backslash \{\overleftarrow{e}\}$, are identically distributed under $\mathbf{Q}$. We denote by β a generic random variable distributed as $\beta(x)$. Since the walk is supposed transient, $\beta > 0$ $\mathbf{Q}$-almost surely, and in particular $E_{\mathbf{Q}}[\beta] > 0$.

We still consider a general Galton–Watson tree. We prove that the number of sites visited at a generation has a bounded expectation under $\mathbb{Q}$.

Lemma 2.1. *There exists a constant c_1 such that for any $n \geq 0$,*

$$E_{\mathbb{Q}}\left[\sum_{|x|=n} \mathbb{1}_{\{T_x < \infty\}}\right] \leq c_1.$$

Proof. By the Markov property, for any $n \geq 0$,

$$\sum_{|x|=n} P_\omega^e(T_x < \infty)\beta(x) = \sum_{|x|=n} P_\omega^e(T_x < \infty,\ X_k \neq \overleftarrow{x}\ \forall k > T_x) \leq 1.$$

The last inequality is due to the fact that there is at most one regeneration time at the n-th generation. Since $P_\omega^e(T_x < \infty)$ is independent of $\beta(x)$, we obtain :

$$1 \geq E_{\mathbf{Q}}\left[\sum_{|x|=n} P_\omega^e(T_x < \infty)\beta(x)\right] = \sum_{|x|=n} E_{\mathbf{Q}}\left[P_\omega^e(T_x < \infty)\right] E_{\mathbf{Q}}[\beta].$$

In view of the identity $E_{\mathbb{Q}}\left[\sum_{|x|=n} \mathbb{1}_{\{T_x<\infty\}}\right] = \sum_{|x|=n} E_{\mathbf{Q}}\left[P_\omega^e(T_x < \infty)\right]$, the lemma follows immediately. $\square$

Let us now deal with the case of the regular tree. We suppose in the rest of the section that there exists $b \geq 2$ such that $\nu(x) = b$ for any $x \in \mathbb{T} \setminus \{\overleftarrow{e}\}$.

Lemma 2.2. *If* $\mathbf{E}\left[\frac{1}{\sum_{i=1}^{b} A_i}\right] < \infty$, *then*

$$\mathbf{E}\left[\frac{1}{\beta}\right] < \infty\,.$$

Proof. Notice that $\mathbf{E}\left[\frac{1}{\max_{1\le i\le b} A_i}\right] < \infty$. For any $n \ge 0$, call v_n the vertex defined by iteration in the following way :

- $v_0 = e$
- $v_n \le v_{n+1}$ and $A(v_{n+1}) = \max\{A(y),\, y \text{ is a child of } v_n\}$.

The Markov property tells that

$$\beta(x) = \sum_{i=1}^{b} \omega(x, x_i)\beta(x_i) + \sum_{i=1}^{b} \omega(x, x_i)(1 - \beta(x_i))\beta(x)\,,$$

from which it follows that for any vertex x,

$$\frac{1}{\beta(x)} = 1 + \frac{1}{\sum_{i=1}^{b} A(x_i)\beta(x_i)} \le 1 + \min_{1\le i\le b} \frac{1}{A(x_i)\beta(x_i)}\,. \tag{2.1}$$

Let $\mathcal{C}(v_n) := \{y \text{ is a child of } v_n,\, y \neq v_{n+1}\}$ be the set of children of v_n different from v_{n+1}. Take $C > 0$ and define for any $n \ge 1$ the event

$$E_n := \{\forall k \in [0, n-1]\,, \forall y \in \mathcal{C}(v_k)\,, (A(y)\beta(y))^{-1} > C\}\,.$$

We extend the definition to $n = 0$ by $E_0^c := \emptyset$. Notice that the sequence of events is decreasing. Using equation (2.1) yields

$$\frac{\mathbb{1}_{E_n}}{\beta(v_n)} \le (1 + C) + \frac{\mathbb{1}_{E_{n+1}}}{A(v_{n+1})\beta(v_{n+1})}\,. \tag{2.2}$$

On the other hand, by the i.i.d. property of the environment, we have

$$\mathbf{P}(E_n) = \mathbf{P}(E_1)^n\,.$$

By choosing C such that $\mathbf{P}(E_1) < 1$ and using the Borel–Cantelli lemma, we have $\mathbb{1}_{E_n} = 0$ from some $n_0 \ge 0$ almost surely. Iterate equation (2.2) to obtain

$$\frac{1}{\beta(e)} \le (1 + C)\left(1 + \sum_{n\ge 1} B(n)\right)$$

where $B(n) = 1\!\!1_{E_n} \prod_{k=1}^{n} \frac{1}{A(v_k)}$. Hence

$$\mathbf{E}\left[\frac{1}{\beta}\right] \le (1+C)\left(1+\sum_{n\ge 1}\mathbf{E}\left[B(n)\right]\right) .$$

We observe that $\mathbf{E}[B(n)] = \{\mathbf{E}\left[1\!\!1_{E_1} A(v_1)^{-1}\right]\}^n$. When C tends to infinity, $\mathbf{E}\left[1\!\!1_{E_1} A(v_1)^{-1}\right]$ tends to zero since $\mathbf{E}[A(v_1)^{-1}] < \infty$. Choose C such that $\mathbf{E}\left[1\!\!1_{E_1} A(v_1)^{-1}\right] < 1$ to complete the proof. $\square$

For $x \in \mathbb{T}$ and $n \ge -1$, let

$$\begin{aligned} N(x) &:= \sum_{k\ge 0} 1\!\!1_{\{X_k = x\}} , \\ N_n &:= \sum_{|x|=n} N(x) , \\ \tau_n &:= \inf\{k \ge 0 \,:\, |X_k| = n\} . \end{aligned}$$

In words, $N(x)$ and N_n denote, respectively, the time spent by the walk at x and at the n-th generation, and τ_n stands for the first time the walk reaches the n-th generation. A consequence of the law of large numbers is that

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \frac{1}{v} \quad \mathbb{Q}\text{-}a.s.$$

Our next result gives an upper bound for the expected value of N_n.

Proposition 2.3. *Suppose that $\mathbf{E}\left[\frac{1}{\sum_{i=1}^{b} A(x_i)}\right] < \infty$. There exists a constant c_2 such that for all $n \ge 0$, we have*

$$\mathbb{E}\left[\sum_{k=0}^{n} N_k\right] \le c_2\, n .$$

Proof. By the strong Markov property, $P_\omega^x(N(x) = \ell) = \{P_\omega^x(T_x^* < \infty)\}^{\ell-1} P_\omega^x(T_x^* = \infty)$, for $\ell \ge 1$. Accordingly,

$$E_\omega^e\left[\sum_{k=0}^{n} N_k\right] = \sum_{0\le|x|\le n} P_\omega^e(T_x < \infty) E_\omega^x[N(x)] = \sum_{0\le|x|\le n} \frac{P_\omega^e(T_x < \infty)}{1 - P_\omega^x(T_x^* < \infty)} .$$

We observe that $1 - P_\omega^x(T_x^* < \infty) \ge \sum_{i=1}^{b} \omega(x, x_i)\, \beta(x_i)$. Since $P_\omega^e(T_x < \infty)$ is independent of $(\omega(x, x_i)\beta(x_i),\ 1 \le i \le b)$, we have

$$\mathbb{E}\left[\sum_{k=0}^{n} N_k\right] \;\le\; \sum_{0\le|x|\le n} \mathbf{E}\left[P_\omega^e(T_x < \infty)\right] \mathbf{E}\left[\left(\sum_{i=1}^{b} \omega(e, e_i)\beta(e_i)\right)^{-1}\right]$$

$$(2.3) \qquad = \mathbf{E}\left[\sum_{0\le|x|\le n} P_\omega^e(T_x<\infty)\right]\mathbf{E}\left[\left(\sum_{i=1}^b \omega(e,e_i)\beta(e_i)\right)^{-1}\right].$$

Since $\sum_{i=1}^b \omega(e,e_i)\,\beta(e_i) \ \ge\ \{\min_{i=1\dots b}\beta(e_i)\}\sum_{i=1}^b \omega(e,e_i)$, it follows that

$$\mathbb{E}\left[\sum_{k=0}^n N_k\right] \le \mathbf{E}\left[\sum_{0\le|x|\le n} P_\omega^e(T_x<\infty)\right]\mathbf{E}\left[\frac{1}{1-\omega(e,\overleftarrow{e})}\right]\mathbf{E}\left[\left(\min_{i=1\dots b}\beta(e_i)\right)^{-1}\right].$$

By definition, $\frac{1}{1-\omega(e,\overleftarrow{e})} = 1+\frac{1}{\sum_{i=1}^b A(e_i)}$, which implies that $\mathbf{E}\left[\frac{1}{1-\omega(e,\overleftarrow{e})}\right]<\infty$. Notice also that $\mathbf{E}\left[(\min_{i=1\dots b}\beta(e_i))^{-1}\right] \le b\mathbf{E}[\frac{1}{\beta}]<\infty$ by Lemma 2.2. Finally, use Lemma 2.1 to complete the proof. $\square$

We are now able to prove the positivity of the speed.

Proof of Theorem 1.5. We note that $\tau_n \le \sum_{k=-1}^n N_k$ and that $N_{-1}\le N_0$. By Proposition 2.3, we have $\mathbb{E}[\tau_n]\le 2c_2\, n$. Fatou's lemma yields that $\mathbb{E}[\liminf_{n\to\infty}\frac{\tau_n}{n}]\le 2c_2$. Since $\lim_{n\to\infty}\frac{\tau_n}{n} = \frac{1}{v}$, then $v>0$. $\square$

3 Proof of Theorem 1.3 : upper bound

This section is devoted to the proof of the upper bound in Theorem 1.3, which is equivalent to the following :

Proposition 3.1. *We have*

$$\liminf_{n\to\infty}\frac{\ln(\tau_n)}{\ln(n)} \ge \frac{1}{\Lambda} \qquad \mathbb{Q}-a.s.$$

3.1 Basic facts about regeneration times

We recall some basic facts about regeneration times for the transient RWRE. These facts can be found in [Gro04] in the case of regular trees, and in [LPP96] in the case of biased random walks on Galton–Watson trees.

Let

$$D(x) := \inf\left\{k\ge 1\ :\ X_{k-1}=x,\ X_k=\overleftarrow{x}\right\}, \quad (\inf\emptyset := \infty).$$

We define the first regeneration time

$$\Gamma_1 := \inf\left\{k > 0 \,:\, \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\right\}$$

as the first time when the walk reaches a generation by a vertex having more than two children and never returns to its parent. We define by iteration

$$\Gamma_n := \inf\left\{k > \Gamma_{n-1} \,:\, \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\right\}$$

for any $n \geq 2$ and we denote by $\mathbb{S}(.)$ the conditional distribution $\mathbb{Q}(.\,|\,\nu(e) \geq 2,\, D(e) = \infty)$.

Fact *Assume that the walk is transient.*

(i) *For any* $n \geq 1$, $\Gamma_n < \infty$ $\quad\mathbb{Q}$*-a.s.*

(ii) *Under* $\mathbb{Q}, (\Gamma_{n+1} - \Gamma_n, |X_{\Gamma_{n+1}}| - |X_{\Gamma_n}|)$, $n \geq 1$ *are independent and distributed as* $(\Gamma_1, |X_{\Gamma_1}|)$ *under the distribution* $\mathbb{S}$.

(iii) *We have* $E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,] < \infty$.

(iv) *The speed verifies* $v = \frac{E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,]}{E_{\mathbb{S}}[\,\Gamma_1\,]}$.

We feel free to omit the proofs of (i),(ii) and (iv), since they easily follow the lines in [Gro04] and [LPP96]. To prove (iii), we will show that $E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,] = 1/E_{\mathbf{Q}}[\beta]$. For any $n \geq 0$, we have, conditionally on $|X_{\Gamma_1}|$,

$$\mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| = n \,\middle|\, |X_{\Gamma_1}|\right) = \mathbb{1}_{\{|X_{\Gamma_1}| \leq n\}}\mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| - |X_{\Gamma_1}| = n - |X_{\Gamma_1}| \,\middle|\, |X_{\Gamma_1}|\right).$$

By the renewal theorem (see chapter XI of [Fel71] for instance) and the fact that $\mathbb{1}_{\{|X_{\Gamma_1}| \leq n\}}$ tends to 1 $\mathbb{Q}$-almost surely, we obtain that

$$\lim_{n\to\infty} \mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| = n \,\middle|\, |X_{\Gamma_1}|\right) = 1/E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,].$$

The dominated convergence yields then

$$\lim_{n\to\infty} \mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| = n\right) = 1/E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,].$$

It remains to notice that on the other hand,

$$\mathbb{Q}\left(\exists k \in \mathbb{N} \,:\, |X_{\Gamma_k}| = n\right) = \mathbb{Q}\left(D(X_{\tau_n}) = \infty\right) = E_{\mathbf{Q}}[\beta]\,. \;\square$$

If we denote for any $n \geq 0$ by $u(n)$ the unique integer such that $\Gamma_{u(n)} \leq \tau_n < \Gamma_{u(n)+1}$, then Fact yields that $\lim_{n\to\infty} \frac{n}{u(n)} = E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,]$. In turn, we deduce that

$$\liminf_{n\to\infty} \frac{\ln(\tau_n)}{\ln(n)} \;\geq\; \liminf_{n\to\infty} \frac{\ln(\Gamma_n)}{\ln(n)} \qquad \mathbb{Q}\text{-}a.s. \tag{3.1}$$

Let for $\lambda \in [0,1]$ and $n \geq 0$,

$$S(n,\lambda) := \sum_{k=1}^{n} (\Gamma_k - \Gamma_{k-1})^{\lambda},$$

by taking $\Gamma_0 := 0$. Then $(\Gamma_n)^{\lambda} \leq S(n,\lambda)$ since $\lambda \leq 1$, which gives, by the law of large numbers,

$$\limsup_{n\to\infty} \frac{(\Gamma_n)^{\lambda}}{n} \leq \lim_{n\to\infty} \frac{S(n,\lambda)}{n} = E_{\mathbb{S}}[\Gamma_1^{\lambda}] \qquad \mathbb{Q}\text{-}a.s. \tag{3.2}$$

3.2 Proof of Proposition 3.1

We construct a RWRE on the half-line as follows; suppose that $\mathbb{T} = \{-1, 0, 1, \ldots\}$. This would correspond to the case where $q_1 = 1$, $e = 0$, $\overleftarrow{e} = -1$. Marking each integer $i \geq 0$ with i.i.d. random variables $A(i)$, we thus define a one-dimensional RWRE as we defined it in the case of a Galton–Watson tree. We call $(R_n)_{n\geq 0}$ this RWRE. We still use the notation P_{ω}^{i} and $\mathbb{P}^i$ to name the quenched and the annealed distribution of (R_n) with $R_0 = i$. For $i \geq -1$ and $a \in \mathbb{R}_+$, define $T_i := \inf\{n \geq 0 \,:\, R_n = i\}$ and

$$p\,(i,a) := \mathbb{P}^0(T_{-1} \wedge T_i > a)\,, \tag{3.3}$$

where $b \wedge c := \min\{b,\, c\}$. We give two preliminary results.

Lemma 3.2. *Let Λ be as in (1.1). Then*

$$\liminf_{a\to\infty} \left\{ \sup_{i\geq 0} \frac{\ln\left(q_1^i p\,(i,a)\right)}{\ln(a)} \right\} \geq -\Lambda\,.$$

Proof. See Section 8. □

We return to our general RWRE $(X_n)_{n\geq 0}$ on a general Galton–Watson tree $\mathbb{T}$.

Lemma 3.3. *We have*

$$\liminf_{a\to\infty} \frac{\ln(\,\mathbb{S}\,(\Gamma_1 > a)\,)}{\ln(a)} \geq -\Lambda\,.$$

Proof. For any $x \in \mathbb{T}$, let $h(x)$ be the unique vertex such that

$$x \leq h(x), \qquad \nu(h(x)) \geq 2\,, \qquad \forall\, y \in \mathbb{T},\, x \leq y < h(x) \Rightarrow \nu(y) = 1\,.$$

In words, $h(x)$ is the oldest descendent of x such that $\nu(h(x)) \geq 2$ (and can be x itself if $\nu(x) \geq 2$). We observe that $\Gamma_1 \geq T_e^* \wedge T_{h(X_1)}$. Moreover, $\{\nu(e) \geq 2,\, D(e) = \infty\} \supset E_1 \cup E_2$ where

$$\begin{aligned} E_1 &:= \{\nu(e) \geq 2\} \cap \left\{X_1 \neq \overleftarrow{e},\, T_e^* < T_{h(X_1)},\, X_{T_e^*+1} \notin \{\overleftarrow{e},\, X_1\}\right\} \cap \{X_n \neq e,\, \forall\, n \geq T_e^* + 1\}\ , \\ E_2 &:= \{\nu(e) \geq 2\} \cap \left\{X_1 \neq \overleftarrow{e},\, T_{h(X_1)} < T_e^*\right\} \cap \left\{X_n \neq \overleftarrow{h(X_1)},\, \forall\, n \geq T_{h(X_1)} + 1\right\}. \end{aligned}$$

It follows that

$$\mathbb{S}(\Gamma_1 > a) \geq \frac{1}{\mathbb{Q}(\nu(e) \geq 2,\, D(e) = \infty)} (\mathbb{Q}(T_e^* > a,\, E_1) + \mathbb{Q}(T_{h(X_1)} > a,\, E_2))\,. \tag{3.4}$$

We claim that

$$\mathbb{Q}(T_e^* > a,\, E_1) = c_3\, \mathbb{Q}(T_{\overleftarrow{e}} < T_{h(e)},\, 1 + T_{\overleftarrow{e}} > a)\,. \tag{3.5}$$

Indeed, write

$$P_\omega^e(T_e^* > a,\, E_1) = \sum_{e_i \neq e_j} P_\omega^e\left(T_e^* < T_{h(e_i)},\, X_1 = e_i,\, X_{T_e^*+1} = e_j,\, D(e_j) = \infty,\, T_e^* > a\right).$$

By gradually applying the strong Markov property at times $T_e^* + 1$, T_e^* and at time 1, this yields

$$P_\omega^e(T_e^* > a,\, E_1) = \sum_{e_i \neq e_j} \omega(e, e_i) P_\omega^{e_i}\left(T_e < T_{h(e_i)},\, 1 + T_e > a\right) \omega(e, e_j)\beta(e_j).$$

Since $\omega(e, e_i)\omega(e_j)$, $\beta(e_j)$ and $P_\omega^{e_i}\left(T_e < T_{h(e_i)},\, 1 + T_e > a\right)$ are independent under $\mathbf{P}$, this leads to

$$\mathbb{P}(T_e^* > a,\, E_1) = \sum_{e_i \neq e_j} \mathbf{E}\left[\omega(e, e_i)\omega(e, e_j)\right] \mathbb{P}^{e_i}\left(T_e < T_{h(e_i)},\, 1 + T_e > a\right) \mathbf{E}\left[\beta(e_j)\right].$$

By the Galton–Watson property,

$$\mathbb{Q}(T_e^* > a,\, E_1) = E_{\mathbf{Q}}\left[\mathbb{1}_{\{\nu(e)\geq 2\}} \sum_{e_i \neq e_j} \omega(e, e_i)\omega(e, e_j)\right] \mathbb{Q}^e\left(T_{\overleftarrow{e}} < T_{h(e)},\, 1 + T_{\overleftarrow{e}} > a\right) E_{\mathbf{Q}}\left[\beta\right]\ ,$$

which gives (3.5). Similarly,

$$\mathbb{Q}(T_{h(X_1)} > a,\, E_2) = c_4 \mathbb{Q}\left(T_{\overleftarrow{e}} > T_{h(e)},\, 1 + T_{h(e)} > a\right)\ . \tag{3.6}$$

Finally, by (3.4), (3.5) and (3.6) we get

$$\mathbb{S}(\Gamma_1 > a) \geq c_5\, \mathbb{Q}\left(1 + T_{\overleftarrow{e}} \wedge T_{h(e)} > a\right).$$

Conditionally on $|\, h(e)|$, the walk $|\, X_n|$, $0 \leq n \leq T_{\overleftarrow{e}} \wedge T_{h(e)}$ has the distribution of the walk R_n, $0 \leq n \leq T_{-1} \wedge T_{|h(e)|}$, as defined at the beginning of this section. For any $n \geq 0$, since $GW(|h(e)| = n) = q_1^n(1-q_1)$, it follows that $\mathbb{Q}\left(1 + T_{\overleftarrow{e}} \wedge T_{h(e)} > a\right) \geq q_1^n(1-q_1)p\,(n,a)$. Finally,

$$\liminf_{a\to\infty} \frac{\ln\left(\mathbb{S}\left(\Gamma_1 > a\right)\right)}{\ln(a)} \geq \liminf_{a\to\infty} \left\{\sup_{n\geq 0} \frac{\ln\left(q_1^n p\,(n,a)\right)}{\ln(a)}\right\}.$$

Applying Lemma 3.2 completes the proof. □

We now have all of the ingredients needed for the proof of Proposition 3.1.

Proof of Proposition 3.1. If $\Lambda \geq 1$, Proposition 3.1 trivially holds since $\tau_n \geq n$. We suppose that $\Lambda < 1$, and let $\Lambda < \lambda < 1$. Let $M_n := \max\{\Gamma_k - \Gamma_{k-1},\ k = 2, \dots n\}$. We have $\mathbb{Q}\left(M_n \leq n^{\frac{1}{\lambda}}\right) = \mathbb{Q}\left(\Gamma_2 - \Gamma_1 \leq n^{\frac{1}{\lambda}}\right)^n$. By Lemma 3.3, $\mathbb{Q}\left(\Gamma_2 - \Gamma_1 \leq n^{\frac{1}{\lambda}}\right) \leq 1 - n^{-1+\varepsilon}$ for some $\varepsilon > 0$ and large n. Consequently, $\sum_{n\geq 1} \mathbb{Q}\left(M_n \leq n^{\frac{1}{\lambda}}\right) < \infty$, and the Borel-Cantelli lemma tells that $\mathbb{Q}$-almost surely and for sufficiently large n, $M_n \geq n^{\frac{1}{\lambda}}$, which in turn implies that $\liminf_{n\to\infty} \frac{\Gamma_n - \Gamma_1}{n^{\frac{1}{\lambda}}} \geq 1$. We proved then that $\liminf_{n\to\infty} \frac{\ln(\Gamma_n)}{\ln(n)} \geq \frac{1}{\Lambda}$. Therefore, by equation (3.1),

$$\liminf_{n\to\infty} \frac{\ln(\tau_n)}{\ln(n)} \geq \frac{1}{\Lambda} \qquad \mathbb{Q}\text{-}\,a.s.\ \square$$

4 Technical results

We give, in this section, some tools needed in our proof of the lower bound in Theorem 1.3. Z_n stands as before for the size of the n-th generation of $\mathbb{T}$.

Lemma 4.1. *For every $b, n \geq 1$, we have*

$$E_{GW}[Z_n \mathbb{1}_{\{Z_n \leq b\}}] \leq b n^b q_1^{n-b}.$$

Proof. If $Z_n \leq b$, then there are at most b vertices before the n-th generation having more than one child. Therefore,

$$GW(Z_n \leq b) \leq C_n^b q_1^{n-b} \leq n^b q_1^{n-b}$$

and we conclude since $E_{GW}[Z_n \mathbb{1}_{\{Z_n \leq b\}}] \leq b\, GW(Z_n \leq b)$. □

Lemma 4.2. *Let β_i, $i \geq 1$ be independent random variables distributed as β. There exists $b_0 \geq 1$ such that*

$$E_{\mathbf{Q}}\left[\left(\frac{1}{\sum_{i=1}^{b_0}\beta_i}\right)^2\right] < \infty\,.$$

Proof. Let $\mathbb{T}^{(i)}$, $i \geq 1$ be independent Galton–Watson trees of distribution GW. We equip independently each $\mathbb{T}^{(i)}$ with an environment of distribution $\mathbf{P}$ so that we can look at the random variable $\beta(e^{(i)})$ where $e^{(i)}$ is the root of $\mathbb{T}^{(i)}$. Then $\beta(e^{(i)})$, $i \geq 1$ are independent random variables distributed as β.

Let $c_6 > 0$ be such that $\eta := \mathbf{Q}(\frac{1}{\beta} > c_6) < 1$. Recall that $\frac{1}{\alpha} \leq A(x) \leq \alpha$, $\forall x \in \mathbb{T}$, $\mathbb{Q}$-almost surely. Let $R^{(i)} := \inf\{n \geq 0 : \exists y \in \mathbb{T}^{(i)},\, |y| = n,\, \frac{1}{\beta(y)} \leq c_6\}$ be the first generation in $\mathbb{T}^{(i)}$ where a vertex verifies $\frac{1}{\beta(y)} \leq c_6$, and let $y^{(i)}$ be such a vertex y. Recall from equation (2.1) that

$$\frac{1}{\beta(x)} \leq 1 + \frac{1}{A(x_j)\beta(x_j)}$$

for any child x_j of a vertex x. By iterating the inequality on the path $[\![e^{(i)}, y^{(i)}]\!]$, we obtain

$$\frac{1}{\beta(e^{(i)})} \leq 1 + \sum_{z \in]\!]e, y^{(i)}[\![} H(z) + \frac{H(y^{(i)})}{\beta(y^{(i)})}$$

where $H(z) = \prod_{v \in]\!]e^{(i)}, z]\!]} \frac{1}{A(v)} \leq \alpha^{|z|}$ for every $z \in \mathbb{T}$ by the bound assumption on A. Since $\frac{1}{\beta(y^{(i)})} \leq c_6$, this implies

$$\frac{1}{\beta(e^{(i)})} \leq c_7\, \alpha^{R^{(i)}}\,,$$

for some constant c_7. There exist constants c_8 and c_9 such that for any $b \geq 1$,

$$\left(\frac{1}{\sum_{i=1}^{b}\beta(e^{(i)})}\right)^2 \leq c_8\, c_9^{\min_{1\leq i\leq b} R^{(i)}}\,. \tag{4.1}$$

We observe that

$$\begin{aligned} E_{\mathbf{Q}}\left[c_9^{\min_{1\leq i\leq b} R^{(i)}}\right] &= \sum_{n=0}^{\infty} c_9^n\, \mathbf{Q}(\min_{1\leq i\leq b} R^{(i)} = n) \\ &\leq \sum_{n=0}^{\infty} c_9^n\, \mathbf{Q}(R^{(1)} \geq n)^b\,. \end{aligned} \tag{4.2}$$

We have, for any $n \geq 1$, $\mathbf{Q}(R^{(1)} \geq n) \leq \mathbf{Q}\left(\forall |x| = n-1, \frac{1}{\beta(x)} > c_6\right)$. Recall that $\eta := \mathbf{Q}(\frac{1}{\beta} > c_6) < 1$. By independence,

$$\mathbf{Q}\left(\forall |x| = n-1, \frac{1}{\beta(x)} > c_6\right) = E_{GW}[\eta^{Z_{n-1}}].$$

Let $q_1 < a < 1$. There exists a constant c_{10} such that $E_{GW}\left[\eta^{Z_\ell}\right] \leq c_{10}\, a^{\ell+1}$ for any $\ell \geq 0$. Choose b_0 such that $c_9 a^{b_0} < 1$. Then by (4.2), $E_{\mathbb{Q}}\left[c_9^{\min_{1\leq i\leq b_0} R^{(i)}}\right] < \infty$, which completes the proof in view of (4.1). □

Define for any $u, v \in \mathbb{T}$ such that $u \leq v$ and for any $n \geq 1$:

$$p_1(u,v) = P_\omega^u\left(T_{\overleftarrow{u}} = \infty\,,\, T_u^* = \infty\,,\, T_v = \infty\right)\,, \tag{4.3}$$

$$\nu(u,n) = \#\left\{x \in \mathbb{T}\,:\, u \leq x, |\,x-u| = n\right\}\,. \tag{4.4}$$

Lemma 4.3. *For all $n \geq 2$ and $k \in \{1,2\}$, we have*

$$E_{\mathbf{Q}}\left[\sum_{|u|=n} \frac{\mathbb{1}_{\{Z_n > b_0\}}}{[p_1(e,u)]^k}\right] < \infty\,. \tag{4.5}$$

Proof. Let $n \geq 2$ and $k \in \{1,2\}$ be fixed integers and $\widetilde{n} := \inf\{\ell \geq 1 \;:\; Z_\ell > b_0\}$. Notice that $\{Z_n > b_0\} = \{\widetilde{n} \leq n\}$. For any $u \in \mathbb{T}$ such that $|u| \geq \widetilde{n}$, let $\widetilde{u} \in \mathbb{T}$ be the unique vertex such that $|\widetilde{u}| = \widetilde{n}$ and $\widetilde{u} \leq u$ that is the ancestor of u at generation $\widetilde{n}$. We have by the Markov property,

$$p_1(e,u) \geq \sum_{|y|=\widetilde{n}-1} P_\omega^e(T_y < T_e*) P_\omega^y(T_{\overleftarrow{y}} = \infty\,,\, T_{\widetilde{u}} = \infty). \tag{4.6}$$

For any $|y| \leq \widetilde{n}$ and y_i child of y, we observe that

$$\omega(y, y_i) = \frac{A(y_i)}{1 + \sum_{j=1}^{\nu(y)} A(y_j)} \geq \frac{1}{c_{11}\nu(y)}\,,$$

which is greater than $1/c_{11}b_0 := c_{12}$, by the boundedness assumption on A and the definition of $\widetilde{n}$. It yields that for any $|y| = \widetilde{n}-1$,

$$P_\omega^e(T_y < T_e^*) \geq P_\omega^e(X_{\widetilde{n}-1} = y) \geq c_{12}^{\widetilde{n}}\,. \tag{4.7}$$

By the Markov property,

$$\begin{aligned} & P^y_\omega(T_{\overleftarrow{y}} = \infty, T_{y_i} = \infty) \\ = \ & \sum_{j\neq i} \omega(y,y_j)\beta(y_j) + \left(\sum_{j\neq i} \omega(y,y_j)(1-\beta(y_j)) \right) P^y_\omega(T_{\overleftarrow{y}} = \infty, T_{y_i} = \infty). \end{aligned}$$

This leads to

$$\begin{aligned} P^y_\omega(T_{\overleftarrow{y}} = \infty, T_{y_i} = \infty) &= \frac{\sum_{j\neq i} A(y_j)\beta(y_j)}{1 + A(y_i) + \sum_{j\neq i} A(y_j)\beta(y_j)} \\ &\geq \frac{1}{\alpha(1+\alpha)} \frac{\sum_{j\neq i}\beta(y_j)}{1+\sum_{j\neq i}\beta(y_j)} \\ &\geq \frac{1}{2\alpha(1+\alpha)} \left(1 \wedge \sum_{j\neq i} \beta(y_j) \right). \end{aligned}$$

Similarly, $P^y_\omega(T_{\overleftarrow{y}} = \infty) \geq \frac{1}{2\alpha^2}\left(1 \wedge \sum_{j=1}^{\nu(y)} \beta(y_j)\right)$. Thus, we have for any $|y| = \widetilde{n} - 1$,

(4.8)
$$P^y_\omega(T_{\overleftarrow{y}} = \infty\, ,\, T_{\widetilde{u}} = \infty) \geq c_{13} \left(1 \wedge \sum_{y_j \neq \widetilde{u}} \beta(y_j) \right).$$

By equations (4.6), (4.7) and (4.8), we have

$$p_1(e,u) \geq c_{13} c_{12}^{\widetilde{n}} \left(1 \wedge \sum_{|x|=\widetilde{n}: x\neq \widetilde{u}} \beta(x) \right).$$

Therefore, arguing over the value of $\widetilde{u}$, we obtain

$$\mathbb{1}_{\{n\geq \tilde{n}\}} \sum_{|u|=n} \mathbf{E}\left[\frac{1}{[p_1(e,u)]^k} \right] \leq c_{14} \sum_{|y|=\widetilde{n}} \nu(y, n-\tilde{n}) \mathbf{E}\left[1 \vee \frac{1}{[\sum_{|x|=\tilde{n}, x\neq y} \beta(x)]^k} \right],$$

where $c_{14} := (c_{13}c_{12}^n)^{-k}$. By using the Galton–Watson property at generation $\widetilde{n}$,

$$\begin{aligned} & \sum_{|u|=n} E_{\mathbf{Q}} \left[\frac{\mathbb{1}_{\{u\in\mathbb{T}, Z_n > b_0\}}}{[p_1(e,u)]^k} \,\Big|\, \widetilde{n}\, , Z_0, \ldots, Z_{\widetilde{n}} \right] \\ \leq \ & c_{14} \sum_{|y|=\widetilde{n}} E_{GW}[\nu(y, n-\widetilde{n})] E_{\mathbf{Q}} \left[1 \vee \frac{1}{[\sum_{i=1}^p \beta(i)]^k} \right]_{p = Z_{\widetilde{n}}-1} \\ \leq \ & c_{15} Z_{\widetilde{n}} \end{aligned}$$

by Lemma 4.2. Integrating over GW completes the proof of (4.5). $\square$

Remark. Lemma 4.3 tells in particular that

$$E_{\mathbf{Q}}\left[\frac{\mathbb{1}_{\{Z_n>b_0\}}}{\beta(e)}\right] \leq E_{\mathbf{Q}}\left[\frac{\mathbb{1}_{\{Z_n>b_0\}}}{P^e_\omega(T_{\overleftarrow{e}}=\infty, T^*_e=\infty)}\right] < \infty\,. \tag{4.9}$$

We deal now with a comparison between RWREs on a tree and one-dimensional RWREs already used in [LPP96]. Let $\mathbb{T}$ be a tree and ω the environment on this tree. Take $x \leq y \in \mathbb{T}$. We look at the path $[\![\overleftarrow{x}, y]\!] = \{\overleftarrow{x} = x_{-1}, x_0, \ldots, x_p = y\}$ defined as the shortest path from $\overleftarrow{x}$ to y, and we consider on it the random walk $(\widetilde{X}_n)$ with probability transitions $\widetilde{\omega}(\overleftarrow{x}, x) = \widetilde{\omega}(y, x_{p-1}) = 1$ and for any $0 \leq i < p$,

$$\begin{aligned}
\widetilde{\omega}(x_i, x_{i+1}) &= \frac{\omega(x_i, x_{i+1})}{\omega(x_i, x_{i+1}) + \omega(x_i, x_i - 1)}\,,\\
\widetilde{\omega}(x_i, x_{i-1}) &= \frac{\omega(x_i, x_{i-1})}{\omega(x_i, x_{i+1}) + \omega(x_i, x_{i-1})}\,.
\end{aligned}$$

Thus we can associate to the pair (x, y) a one-dimensional RWRE on $[\![\overleftarrow{x}, y]\!]$, and we denote by $\widetilde{P}$, $\widetilde{E}$ the probabilities and expectations related to this new RWRE. We observe that under $\mathbb{Q}^x$, the RWRE $(\widetilde{X}_n,\, n \leq T_{\overleftarrow{x}} \wedge T_y)$ has the distribution of the RWRE $(R_n,\, n \leq T_{-1} \wedge T_p)$ introduced in Section 3.2. For any $x, y \in \mathbb{T}$, the event $\{T_x < T_y\}$ means that $T_x < \infty$ and $T_x < T_y$.

Lemma 4.4. *For any $x, y, z \in \mathbb{T}$ with $x \leq z < y$,*

$$\begin{aligned}
P^z_\omega(T_y < T_{\overleftarrow{x}}) &\leq \widetilde{P}^z_\omega(T_y < T_{\overleftarrow{x}})\,,\\
P^z_\omega(T_{\overleftarrow{x}} < T_y) &\leq \widetilde{P}^z_\omega(T_{\overleftarrow{x}} < T_y)\,.
\end{aligned}$$

Proof. Fix $z_1, \ldots z_{n-1}$ in $]\!]\overleftarrow{x}, y[\![$ and $z_n \in [\![\overleftarrow{x}, y]\!]$. Then

$$P^z_\omega(X_1 = z_1, \ldots, X_n = z_n) = \frac{\omega(z, z_1)}{1 - f(z)} \cdots \frac{\omega(z_{n-1}, z_n)}{1 - f(z_{n-1})}$$

where $f(r)$ represents the probability of making an excursion away from the path $[\![\overleftarrow{x}, y]\!]$ from the vertex r. For each $r \in [\![\overleftarrow{x}, y[\![$, call r^+ the child of r which lies in the path. Then $f(r) \leq 1 - \omega(r, r^+) - \omega(r, \overleftarrow{r})$. It follows that

$$\begin{aligned}
P^z_\omega(X_1 = z_1, \ldots, X_n = z_n) &\leq \widetilde{\omega}(z, z_1) \ldots \widetilde{\omega}(z_{n-1}, z_n)\\
&= \widetilde{P}^z_\omega(\widetilde{X}_1 = z_1, \ldots, \widetilde{X}_n = z_n)\,.
\end{aligned}$$

It remains to see that the events $\{T_y < T_x\}$ and $\{T_x < T_y\}$ can be written as an union of disjoint sets of the form $\{X_1 = z_1, \ldots, X_n = z_n\}$. $\square$

The last lemma deals with the one-dimensional RWRE $(R_n)_{n\geq 0}$ defined in Section 3.2.

Lemma 4.5. *For any $n \geq 1$, there exists a number $c_{19}(n)$ such that for any $i > n$ and almost every ω,*

$$E^0_\omega[T_{-1} \wedge T_i] \leq c_{19} E^n_\omega[T_{n-1} \wedge T_i]\,.$$

Proof. Let $i > n \geq 1$. By the Markov property and for $0 < p \leq i$, we have

$$E^{p-1}_\omega[T_{p-2} \wedge T_i] = 1 + \omega(p-1,p)\left\{E^p_\omega[T_{p-1} \wedge T_i] + P^p_\omega(T_{p-1} < T_i) E^{p-1}_\omega[T_{p-2} \wedge T_i]\right\}$$

which gives that $E^{p-1}_\omega[T_{p-2} \wedge T_i] = \frac{1+\omega(p-1,p)E^p_\omega[T_{p-1}\wedge T_i]}{1-\omega(p-1,p)P^p_\omega(T_{p-1}\wedge T_i)}$, so that for some c_{20}, c_{21} and c_{22} we have

$$E^{p-1}_\omega[T_{p-2} \wedge T_i] \leq c_{20} + c_{21} E^p_\omega[T_{p-1} \wedge T_i] \leq c_{22} E^p_\omega[T_{p-1} \wedge T_i].$$

Iterating the inequality over all p from 1 to n gives the desired inequality. $\square$

5 Proof of Theorem 1.3 : lower bound

Let $(R_n)_{n\geq 0}$ be the one-dimensional RWRE associated with $\mathbb{T} = \{-1, 0, 1, \ldots\}$ defined in Section 3.2 and $T_i = \inf\{k \geq 0 :\ R_k = i\}$. Define for any $\lambda \geq 0$,

(5.1) $$m(n,\lambda) := \mathbf{E}\left[\left(E^0_\omega\left[T_{-1} \wedge T_n\right]\right)^\lambda\right]\,,$$

and let

(5.2) $$\lambda_c := \sup\left\{\lambda \geq 0\ :\ \exists r > q_1 \text{ such that } \sum_{n\geq 0} m(n,\lambda) r^n < \infty\right\}.$$

We start with a lemma.

Lemma 5.1. *We have $\Lambda \leq \lambda_c$.*

Proof. See Section 8. $\square$

Take a $\lambda \in [0,1]$ such that $\lambda < \Lambda$. By Lemma 5.1, we have $\lambda < \lambda_c$ which in turn implies by (5.2) that there exists an $1 > r > q_1$ such that

(5.3) $$\sum_{n\geq 0} m(n,\lambda)\,(n+1) r^n < \infty\,.$$

Recall the definition of b_0 in Lemma 4.2. Then, by Lemma 4.1, we can define

$$n_0 := \inf\left\{n \geq 1 \,:\, E_{GW}[Z_n \mathbb{1}_{\{Z_n \leq b_0\}}] \leq r^n\right\} .$$

Let $\mathbb{T}_{n_0}$ be the subtree of $\mathbb{T}$ defined as follows : y is a child of x in $\mathbb{T}_{n_0}$ if $x \leq y$ and $|y-x| = n_0$. In this new Galton–Watson tree $\mathbb{T}_{n_0}$, we define

$$\mathbb{W} = \mathbb{W}(\mathbb{T}) := \{x \in \mathbb{T}_{n_0} : \forall y \in \mathbb{T}_{n_0}, (y < x) \Rightarrow \nu(y, n_0) \leq b_0\} , \tag{5.4}$$

where $\nu(y, n_0)$ is defined in (4.4). We call W_k the size of the k-th generation of $\mathbb{W}$. The subtree $\mathbb{W}$ is a Galton–Watson tree, whose offspring distribution is of mean $E_{GW}[Z_{n_0} \mathbb{1}_{\{Z_{n_0} \leq b_0\}}] \leq r^{n_0}$. In particular, we have for any $k \geq 0$,

$$E_{GW}[W_k] \leq r^{kn_0} . \tag{5.5}$$

For any $y \in \mathbb{T}$, we denote by y_{n_0} the youngest ancestor of y belonging to $\mathbb{T}_{n_0}$, or equivalently the unique vertex such that

$$y_{n_0} \leq y, \qquad y_{n_0} \in \mathbb{T}_{n_0}, \qquad \forall\, z \in \mathbb{T}_{n_0} \;\; z \leq y \Rightarrow z \leq y_{n_0} .$$

Let

$$\begin{aligned} N_{1,n} &:= \sum_{|y|=n} N(y) \mathbb{1}_{\{\nu(y_{n_0}, n_0) > b_0\}} , \\ N_{2,n} &:= \sum_{|y|=n} N(y) \mathbb{1}_{\{\nu(y_{n_0}, n_0) \leq b_0, y_{n_0} \notin \mathbb{W}\}} . \end{aligned}$$

Lemma 5.2. *There exists a constant L such that for any $n \geq n_0$:*

$$E_{\mathbb{Q}}[N_{1,n}] \;\leq\; L , \tag{5.6}$$

$$E_{\mathbb{Q}}[N_{2,n}^{\lambda}] \;\leq\; L . \tag{5.7}$$

We admit Lemma 5.2 for the time being, and show how it implies Theorem 1.3.

Proof of Theorem 1.3 : lower bound. Notice that $\mathbb{W}$ is finite almost surely. Then, there exists a random $K \geq 0$ such that for $n \geq K$, $N_n \leq N_{1,n} + N_{2,n}$. Lemma 5.2 yields that $E_{\mathbb{Q}}[N_n^{\lambda} \mathbb{1}_{\{n \geq K\}}] \leq L^{\lambda} + L$ for any $n \geq n_0$. By Fatou's lemma, $\liminf_{n\to\infty} \frac{\sum_{k=K}^{n} N_k^{\lambda}}{n} < \infty$. Denote by $(r_k,\, k \geq 0)$ the sequence $(|X_{\Gamma_k}|,\, k \geq 0)$. Notice that for any $k \geq 1$,

$$\Gamma_{k+1} - \Gamma_k = \sum_{i=r_k+1}^{r_{k+1}} N_i .$$

It yields that $S(u(n),\lambda) := \sum_{k=1}^{u(n)}(\Gamma_k - \Gamma_{k-1})^\lambda \leq \sum_{i=0}^{\tau_{u(n)}} N_i^\lambda \leq \sum_{i=0}^{n} N_i^\lambda$ where, as in Section 3, $u(n)$ is the unique integer such that $\Gamma_{u(n)} \leq \tau_n < \Gamma_{u(n)+1}$. Observe also that $\frac{n}{u(n)}$ tends to $E_{\mathbb{S}}\left[|X_{\Gamma_1}|\right]$. It follows that

$$\liminf_{n\to\infty} \frac{1}{n} S(n,\lambda) \leq \liminf_{n\to\infty} \frac{1}{u(n)} \sum_{k=K}^{n} N_k^\lambda = E_{\mathbb{S}}\left[|X_{\Gamma_1}|\right] \liminf_{n\to\infty} \frac{1}{n} \sum_{k=K}^{n} N_k^\lambda < \infty\,.$$

Using equation (3.2) implies that $\limsup_{n\to\infty} \frac{(\Gamma_n)^\lambda}{n} < c_{23}$ for some constant c_{23}. We check that $|X_n| \geq \#\{k : \Gamma_k \leq n\}$ which leads to $|X_n| \geq \frac{n^\lambda}{c_{23}}$ for sufficiently large n. Letting λ go to Λ completes the proof. □

The rest of this section is devoted to the proof of Lemma 5.2. For the sake of clarity, the two estimates, (5.6) and (5.7), are proved in distinct parts.

5.1 Proof of Lemma 5.2 : equation (5.6)

For all $y \in \mathbb{T}$, call Y the youngest ancestor of y such that $\nu(Y, n_0) > b_0$. We have

$$E_\omega^e[N(y)] = P_\omega^e(T_y < \infty) E_\omega^y[N(y)] \leq P_\omega^e(T_Y < \infty) E_\omega^y[N(y)]\,.$$

We compute $E_\omega^y[N(y)]$ with a method similar to the one given in [LPP96]. By the Markov property,

$$E_\omega^y[N(y)] = G(y,Y) + P_\omega^y(T_Y < \infty) P_\omega^Y(T_y < \infty) E_\omega^y[N(y)]\,,$$

where $G(y,Y) := E_\omega^y\left[\sum_{k=0}^{T_Y} \mathbb{1}_{\{X_k = y\}}\right]$. When $\nu(y_{n_0}, n_0) > b_0$, there exists a constant $c_{24} > 0$ such that $P_\omega^y(T_y^* > T_Y) \geq c_{24}$. Therefore, in this case $G(y,Y) \leq (c_{24})^{-1} =: c_{25}$. It follows that

$$\begin{aligned} E_\omega^y[N(y)]\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}} &\leq c_{25} \frac{\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}}{1 - P_\omega^Y(T_y < \infty) P_\omega^y(T_Y < \infty)} \\ &\leq c_{25} \frac{\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}}{1 - P_\omega^Y(T_Y^* < \infty)} \\ &\leq c_{25} \frac{\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}}{\gamma(Y)}\,, \end{aligned}$$

where $\gamma(x) := P_\omega^x(T_{\overleftarrow{x}} = \infty, T_x^* = \infty)$. Arguing over the value of Y yields that

$$E_{\mathbb{Q}}[N_{1,n}] \leq c_{25} E_{\mathbf{Q}}\left[\sum_{n-n_0<|z|\leq n} P_\omega^e(T_z < \infty) \frac{\mathbb{1}_{\{\nu(z,n_0)>b_0\}}}{\gamma(z)}\right]$$

$$\begin{aligned} &= c_{25} E_{\mathbf{Q}} \left[\sum_{n-n_0<|z|\le n} P_\omega^e(T_z<\infty) \right] E_{\mathbf{Q}} \left[\frac{1\!\!1_{\{Z_{n_0}>b_0\}}}{\gamma(e)} \right] \\ &\le c_{25} n_0\, c_1\, c_{26}\,, \end{aligned}$$

by Lemma 2.1 and equation (4.9). □

5.2 Proof of Lemma 5.2 : equation (5.7)

For any $y \in \mathbb{T}$ such that $\nu(y_{n_0}, n_0) \le b_0$ and $y_{n_0} \notin \mathbb{W}$, choose $Y_1 = Y_1(y)$, $Y_2 = Y_2(y)$ and $Y_3 = Y_3(y)$, vertices of $\mathbb{T}_{n_0}$, such that

$$\begin{array}{rll} Y_1 < y, & \nu(Y_1, n_0) > b_0, & \forall\, z \in \mathbb{T}_{n_0},\ Y_1 < z \le y \Rightarrow \nu(z, n_0) \le b_0 \\ Y_1 < Y_2 \le y, & & \forall\, z \in \mathbb{T}_{n_0},\ Y_1 < z \le y \Rightarrow Y_2 \le z\,, \\ y \le Y_3, & \nu(Y_3, n_0) > b_0, & \forall\, z \in \mathbb{T}_{n_0},\ y \le z < Y_3 \Rightarrow \nu(z, n_0) \le b_0\,. \end{array}$$

By definition, Y_1 is the youngest ancestor of y in $\mathbb{T}_{n_0}$ such that $\nu(Y_1, n_0) > b_0$ and Y_2 the child of Y_1 in $\mathbb{T}_{n_0}$ which is also an ancestor of y. In the rest of the section, $\widetilde{P}_\omega = \widetilde{P}_\omega(Y_1, Y_3)$ and $\widetilde{E}_\omega = \widetilde{E}_\omega(Y_1, Y_3)$ represent the probability and expectation for the one-dimensional RWRE associated to the path $[\![Y_1, Y_3]\!]$, as seen in Lemma 4.4. They depend then on the pair (Y_1, Y_3), which doesn't appear in the notation for sake of brevity. Define for any $n \ge n_0$,

$$(5.8) \qquad S(n) := E_{\mathbf{Q}} \left[\sum_{|y|=n: Y_1=e} \left[p_1(e, Y_2)^2 \beta(Y_3) \right]^{-1} \left(\widetilde{E}_\omega^{Y_2} [T_{\overleftarrow{Y}_2} \wedge T_{Y_3}] \right)^\lambda \right],$$

where $\overleftarrow{Y}_2$ represents as usual the parent of Y_2 in the tree $\mathbb{T}$ and $p_1(u, v)$ is defined in (4.3).

Lemma 5.3. *There exists a constant c_{27} such that for any $n \ge n_0$,*

$$E_{\mathbb{Q}}[N_{2,n}^\lambda] \le c_{27} \sum_{k \ge n_0} S(k)\,.$$

Proof. We observe that

$$E_\omega^e[N_n^\lambda] = E_\omega^e \left[\left(\sum_{|y|=n} N(y) \right)^\lambda \right] \le E_\omega^e \left[\sum_{|y|=n} N(y)^\lambda \right]$$

since $\lambda \leq 1$. By the Markov property, $E^e_\omega[\sum_{|y|=n} N(y)^\lambda] = \sum_{|y|=n} P^e_\omega(T_y < \infty)E^y_\omega[N(y)^\lambda]$. An application of Jensen's inequality yields that

$$E^e_\omega[N_n^\lambda] \leq \sum_{|y|=n} P^e_\omega(T_y < \infty)\left(E^y_\omega[N(y)]\right)^\lambda . \tag{5.9}$$

Using the Markov property for any $|y| = n$, we get

$$\begin{aligned} & E^y_\omega[N(y)] \\ = \ & G(y, Y_1 \wedge Y_3) + E^y_\omega[N(y)](P^y_\omega(T_{Y_1} < T_{Y_3})P^{Y_1}_\omega(T_y < \infty) + P^y_\omega(T_{Y_3} < T_{Y_1})P^{Y_3}_\omega(T_y < \infty)), \end{aligned}$$

where $G(y, Y_1 \wedge Y_3) := E^y_\omega\left[\sum_{k=0}^{T_{Y_1}\wedge T_{Y_3}} \mathbb{1}_{\{X_k=y\}}\right]$. Accordingly,

$$E^y_\omega[N(y)] = \frac{G(y, Y_1 \wedge Y_3)}{1 - P^y_\omega(T_{Y_1} < T_{Y_3})P^{Y_1}_\omega(T_y < \infty) - P^y_\omega(T_{Y_3} < T_{Y_1})P^{Y_3}_\omega(T_y < \infty)} .$$

Notice that $\left[1 - P^y_\omega(T_{Y_1} < T_{Y_3})P^{Y_1}_\omega(T_y < \infty) - P^y_\omega(T_{Y_3} < T_{Y_1})P^{Y_3}_\omega(T_y < \infty)\right]^{-1}$ is the expected number of times when the walk go from y to Y_1 or Y_3 and then returns to y, which is naturally smaller than $E^y_\omega[N(Y_1) + N(Y_3)]$. We have

$$\begin{aligned} E^y_\omega[N(Y_1)] &= P^y_\omega(T_{Y_1} < \infty)\left[1 - P^{Y_1}_\omega(T^*_{Y_1} < \infty)\right]^{-1} \\ &\leq [p_1(Y_1, Y_2)]^{-1} , \end{aligned}$$

where as before $p_1(Y_1, Y_2) = P^{Y_1}_\omega\left(T_{\overleftarrow{Y}_1} = \infty\, ,\, T^*_{Y_1} = \infty\, ,\, T_{Y_2} = \infty\right)$. Similarly $E^y_\omega[N(Y_3)] \leq [\beta(Y_3)]^{-1}$. We obtain

$$P^e_\omega(T_y < \infty)\left(E^y_\omega[N(y)]\right)^\lambda \leq [p_1(Y_1, Y_2)\beta(Y_3)]^{-1} P^e_\omega(T_y < \infty)\left(G(y, Y_1 \wedge Y_3)\right)^\lambda . \tag{5.10}$$

We deduce from the Markov property that $P^e_\omega(T_y < \infty) = P^e_\omega(T_{Y_1} < \infty)P^{Y_1}_\omega(T_y < \infty)$ and $P^{Y_1}_\omega(T_y < \infty) = G(Y_1, y)P^{Y_1}_\omega(T_y < T^*_{Y_1})$ where $G(Y_1, y) := E^{Y_1}_\omega\left[\sum_{k=0}^{T_y} \mathbb{1}_{\{X_k=Y_1\}}\right]$. By Lemma 4.4, we have $P^{Y_1}_\omega(T_y < T_{\overleftarrow{Y}_1}) \leq \widetilde{P}^{Y_1}_\omega(T_y < T_{\overleftarrow{Y}_1})$. In words, it means that the probability to escape by y is lower for the RWRE on the tree than for the restriction of the walk on $[\![Y_1, y]\!]$. Furthermore $G(Y_1, y) \leq E^{Y_1}_\omega[N(Y_1)] \leq [p_1(Y_1, Y_2)]^{-1}$, so that

$$\begin{aligned} P^e_\omega(T_y < \infty) &\leq P^e_\omega(T_{Y_1} < \infty)\widetilde{P}^{Y_1}_\omega(T_y < T_{\overleftarrow{Y}_1})[p_1(Y_1, Y_2)]^{-1} \\ &\leq P^e_\omega(T_{Y_1} < \infty)\left(\widetilde{P}^{Y_1}_\omega(T_y < T_{\overleftarrow{Y}_1})\right)^\lambda [p_1(Y_1, Y_2)]^{-1} . \end{aligned} \tag{5.11}$$

We observe that

$$G(y, Y_1 \wedge Y_3) = \left[1 - P^y_\omega(T^*_y < T_{Y_1} \wedge T_{Y_3})\right]^{-1} . \tag{5.12}$$

Call y_3 the unique child of y such that $y_3 \le Y_3$. Consequently,

$$\begin{aligned} & P_\omega^y(T_y^* < T_{Y_1} \wedge T_{Y_3}) \\ \le\ & [1-\omega(y,y_3)-\omega(y,\overleftarrow{y})] + \omega(y,\overleftarrow{y}) P_\omega^{\overleftarrow{y}}(T_y < T_{Y_1}) + \omega(y,y_3) P_\omega^{y_3}(T_y < T_{Y_3}) \,. \end{aligned}$$

By Lemma 4.4, we have

$$\begin{aligned} P_\omega^{\overleftarrow{y}}(T_y < T_{Y_1}) &\le \widetilde{P}_\omega^{\overleftarrow{y}}(T_y < T_{Y_1}) \,, \\ P_\omega^{y_3}(T_y < T_{Y_3}) &\le \widetilde{P}_\omega^{y_3}(T_y < T_{Y_3}) \,. \end{aligned}$$

Equation (5.12) becomes $G(y, Y_1 \wedge Y_3) \le (\omega(y,y_3)+\omega(y,\overleftarrow{y}))^{-1} \widetilde{G}(y, Y_1 \wedge Y_3)$ where $\widetilde{G}(y, Y_1 \wedge Y_3)$ stands for the expectation of the number of times the one-dimensional RWRE associated to the pair (Y_1, Y_3) by Lemma 4.4 crosses y before reaching Y_1 or Y_3 when started from y. Since $\nu(y) \le b_0$, there exists a constant c_{28} such that $(\omega(y,\overleftarrow{y}) + \omega(y,y_3))^{-1} \le c_{28}$. It yields

$$G(y, Y_1 \wedge Y_3) \le c_{28}\, \widetilde{G}(y, Y_1 \wedge Y_3) \,. \tag{5.13}$$

Finally, using (5.11), (5.13), and the following inequality,

$$\widetilde{P}_\omega^{Y_1}(T_y < T_{\overleftarrow{Y}_1})\, \widetilde{G}(y, Y_1 \wedge Y_3) \le \widetilde{E}_\omega^{Y_1}[T_{\overleftarrow{Y}_1} \wedge T_{Y_3}] \,,$$

we get

$$P_\omega^e(T_y < \infty)\, (G(y, Y_1 \wedge Y_3))^\lambda \le \frac{c_{28}}{p_1(Y_1,Y_2)} P_\omega^e(T_{Y_1} < \infty) (\widetilde{E}_\omega^{Y_1}[T_{\overleftarrow{Y}_1} \wedge T_{Y_3}])^\lambda \,.$$

By Lemma 4.5, for any $y \in \mathbb{T}$, we have

$$\widetilde{E}_\omega^{Y_1}[T_{\overleftarrow{Y}_1} \wedge T_{Y_3}] \le c_{19}(n_0) \widetilde{E}_\omega^{Y_2}[T_{\overleftarrow{Y}_2} \wedge T_{Y_3}] \,.$$

It follows that

$$P_\omega^e(T_y < \infty)\, (G(y, Y_1 \wedge Y_3))^\lambda \le \frac{c_{28} c_{19}^\lambda}{p_1(Y_1,Y_2)} P_\omega^e(T_{Y_1} < \infty) (\widetilde{E}_\omega^{Y_2}[T_{\overleftarrow{Y}_2} \wedge T_{Y_3}])^\lambda \,. \tag{5.14}$$

In view of equations (5.10) and (5.14), we obtain

$$P_\omega^e(T_y < \infty)\, (E_\omega^y[N(y)])^\lambda \ \le\ c_{29}\, P_\omega^e(T_{Y_1} < \infty) H(Y_1, y, Y_3)$$

where

$$H(Y_1, y, Y_3) := \left[p_1(Y_1,Y_2)^2 \beta(Y_3)\right]^{-1} \left(\widetilde{E}_\omega^{Y_2}[T_{\overleftarrow{Y}_2} \wedge T_{Y_3}]\right)^\lambda \,.$$

By equation (5.9), it implies that

$$E_{\mathbb{Q}}[N_{2,n}^{\lambda}] \leq c_{29}\, E_{\mathbf{Q}}\left[\sum_{|y|=n} P_{\omega}^{e}(T_{Y_1}<\infty)H(Y_1,y,Y_3)\right].$$

Arguing over the value of Y_1 gives

$$\begin{aligned}
E_{\mathbb{Q}}[N_{2,n}^{\lambda}] &\leq c_{29}\, E_{\mathbf{Q}}\left[\sum_{|z|\leq n-n_0} P_{\omega}^{e}(T_z<\infty)\left(\sum_{|y|=n, Y_1=z} H(z,y,Y_3)\right)\right] \\
&= c_{29}\, E_{\mathbf{Q}}\left[\sum_{|z|\leq n-n_0} P_{\omega}^{e}(T_z<\infty)E_{\mathbf{Q}}\left[\sum_{|y|=n-|z|, Y_1=e} H(e,y,Y_3)\right]\right] \\
&= c_{29}\, E_{\mathbf{Q}}\left[\sum_{|z|\leq n-n_0} P_{\omega}^{e}(T_z<\infty)S(n-|z|)\right],
\end{aligned}$$

by equation (5.8). Lemma 2.1 yields that

$$\begin{aligned}
E_{\mathbb{Q}}[N_{2,n}^{\lambda}] &\leq c_1 c_{29} \sum_{k=n_0}^{n} S(k) \\
&\leq c_1 c_{29} \sum_{k\geq n_0} S(k)\,. \ \square
\end{aligned}$$

We call as before $m(n,\lambda) := \mathbf{E}\left[\left(E_{\omega}^{0}\left[T_{-1}\wedge T_n\right]\right)^{\lambda}\right]$ for the one-dimensional RWRE $(R_n)_{n\geq 0}$. The following lemma gives an estimate of $S(n)$.

Lemma 5.4. *There exists a constant c_{30} such that for any $\ell \geq 0$,*

$$S(\ell+n_0) \leq c_{30} \sum_{i\geq \ell} m(i,\lambda) r^i\,.$$

Proof. Let $\ell \geq 0$ and $f(Y_2,Y_3) := \left(\widetilde{E}^{Y_2}[T_{\overleftarrow{Y}_2}\wedge T_{Y_3}]\right)^{\lambda}$. We have

$$\begin{aligned}
S(\ell+n_0) &= E_{\mathbf{Q}}\left[\sum_{|y|=\ell+n_0: Y_1=e} \left[p_1(e,Y_2)^2\beta(Y_3)\right]^{-1} f(Y_2,Y_3)\right] \\
&= E_{\mathbf{Q}}\left[\sum_{|u|=n_0} \left[p_1(e,u)\right]^{-2} \sum_{|y|=\ell+n_0: Y_2=u} f(u,Y_3)\left[\beta(Y_3)\right]^{-1}\right].
\end{aligned}$$

If we call $\mathbb{T}_u$ the subtree of $\mathbb{T}$ rooted in u, we observe that

$$\sum_{|y|=\ell+n_0:Y_2=u} f(u,Y_3)\,[\beta(Y_3)]^{-1} \leq \mathbb{I}_{\{Z_{n_0}>b_0\}} \sum_{|z|\geq \ell+n_0:z\in\mathbb{W}(\mathbb{T}_u)} f(u,z)\,[\beta(z)]^{-1}\,\mathbb{I}_{\{\nu(z,n_0)>b_0\}}\,,$$

where $\mathbb{W}$ was defined in equation (5.4). The Galton–Watson property yields that

$$\begin{aligned} S(\ell+n_0) &\leq E_{\mathbf{Q}}\left[\sum_{|u|=n_0}\frac{\mathbb{I}_{\{Z_{n_0}>b_0\}}}{p_1(e,u)^2}\right] E_{\mathbf{Q}}\left[\sum_{|z|\geq\ell,z\in\mathbb{W}} f(e,z)\,[\beta(z)]^{-1}\,\mathbb{I}_{\{\nu(z,n_0)>b_0\}}\right] \\ &= E_{\mathbf{Q}}\left[\sum_{|u|=n_0}\frac{\mathbb{I}_{\{Z_{n_0}>b_0\}}}{p_1(e,u)^2}\right] E_{\mathbf{Q}}\left[\sum_{|z|\geq\ell,z\in\mathbb{W}} f(e,z)\right] E_{\mathbf{Q}}\left[\frac{\mathbb{I}_{\{Z_{n_0}>b_0\}}}{\beta(e)}\right] \\ &\leq c_{31}E_{\mathbf{Q}}\left[\sum_{|z|\geq\ell,z\in\mathbb{W}} f(e,z)\right]\,, \end{aligned}$$

by Lemma 4.3 and equation (4.9). The proof follows then from

$$\begin{aligned} E_{\mathbf{Q}}\left[\sum_{|z|\geq\ell,z\in\mathbb{W}} f(e,z)\right] &= E_{GW}\left[\sum_{|z|\geq\ell,z\in\mathbb{W}} m(|z|,\lambda)\right] \\ &= \sum_{i:in_0\geq\ell} m(in_0,\lambda)E_{GW}[W_i] \leq \sum_{in_0\geq\ell} m(in_0,\lambda)r^{in_0}\,, \end{aligned}$$

where the last inequality comes from equation (5.5). □

We are now able to prove (5.7).

Proof of Lemma 5.2, equation (5.7). By Lemma 5.3,

$$E_{\mathbb{Q}}[N^{\lambda}_{2,n}] \leq c_{27}\sum_{\ell\geq 0} S(\ell+n_0)\,.$$

Lemma 5.4 tells that

$$\begin{aligned} \sum_{\ell\geq 0} S(\ell+n_0) &\leq c_{30}\sum_{i\geq\ell\geq 0} m(i,\lambda)r^i \\ &= c_{30}\sum_{i\geq 0}(i+1)m(i,\lambda)r^i\,, \end{aligned}$$

which is finite by equation (5.3). □

6 The positivity of the speed : Theorem 1.1

If we suppose that $\Lambda < 1$, then Theorem 1.3 ensures that $\frac{|X_n|}{n}$ tends to 0. Suppose now that $\Lambda > 1$. Take $\lambda = 1$ in the proof of the lower bound of Theorem 1.3 in Section 5 to see that $|X_n| \geq \frac{n}{c_{23}}$ for sufficiently large n, which proves the positivity of the speed in this case. Theorem 1.1 is proved. $\square$

7 Speed of reinforced random walks : Theorem 1.4

When $b \geq 3$, Theorem 1.4 follows immediately from Theorem 1.5. In the rest of this section, we assume that $\mathbb{T}$ is a binary tree. Thanks to the correspondence between RWRE and LERRW mentioned in the introduction, we only have to prove the positivity of the speed for a RWRE on the binary tree such that the density of $\omega(y,z)$ on $(0,1)$ is given by

$$f_0(x) = 1 \qquad \text{if } z = \overleftarrow{y} \tag{7.1}$$

$$f_1(x) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} x^{-1/2}(1-x)^{1/2} \quad \text{if } z \text{ is a child of } y. \tag{7.2}$$

We propose to prove three lemmas before handling the proof of the theorem.

Lemma 7.1. *We have for any* $0 < \delta < 1$,

$$\mathbf{E}\left[\frac{1}{\beta^\delta}\right] < \infty\,.$$

Proof. By equation (2.1), for any $y \in \mathbb{T}$,

$$\begin{aligned}\frac{1}{\beta(y)^\delta} &\leq \left(1 + \min_{i=1,2} \frac{1}{A(y_i)\beta(y_i)}\right)^\delta \\ &\leq 1 + \min_{i=1,2} \frac{1}{A(y_i)^\delta \beta(y_i)^\delta}\,.\end{aligned}$$

Notice that by (7.1),

$$\mathbf{E}\left[\min_{i=1,2} \frac{1}{A(y_i)^\delta}\right] \leq 2^\delta\, \mathbf{E}\left[\left(\frac{1}{A(y_1)+A(y_2)}\right)^\delta\right] = 2^\delta\, \mathbf{E}\left[\left(\frac{\omega(y,\overleftarrow{y})}{1-\omega(y,\overleftarrow{y})}\right)^\delta\right] < \infty\,.$$

The proof is therefore the proof of Lemma 2.2 when replacing $A(y)$ and $\beta(y)$ respectively by $A(y)^\delta$ and $\beta(y)^\delta$. $\square$

Recall that for any $y \in \mathbb{T}$, $\gamma(y) := P_\omega^y(T_{\overleftarrow{y}} = \infty,\, T_y^* = \infty)$.

Lemma 7.2. *There exists $\mu \in (0,1)$ such that for any $\varepsilon \in (0,1)$, we have*

$$\mathbf{E}\left[\left(\frac{\mathbb{I}_{\{\omega(e,\overleftarrow{e})\leq 1-\varepsilon\}}}{\gamma(e)}\right)^{1/\mu}\right] < \infty .$$

Proof. We see that

$$\frac{1}{\gamma(e)} = \frac{1}{\omega(e,e_1)\beta(e_1)+\omega(e,e_2)\beta(e_2)} \leq \min_{i=1,2} \frac{1}{\omega(e,e_i)\beta(e_i)} .$$

Let $\mu \in (0,1)$ and $\varepsilon \in (0,1)$. We compute $\mathbf{P}(\omega(e,\overleftarrow{e}) \leq 1-\varepsilon\,,\, \min_{i=1,2}\left\{[\omega(e,e_i)\beta(e_i)]^{-1/\mu}\right\} > n)$ for $n \in \mathbb{R}_+^*$. We observe that $\{\omega(e,\overleftarrow{e}) \leq 1-\varepsilon\} \subset \{\omega(e,e_1)\geq \varepsilon/2\} \cup \{\omega(e,e_2)\geq\varepsilon/2\}$. By symmetry,

$$\begin{aligned}
& \mathbf{P}\left(\omega(e,\overleftarrow{e}) \leq 1-\varepsilon\,,\, \min_{i=1,2}\left\{[\omega(e,e_i)\beta(e_i)]^{-1/\mu}\right\} > n\right)\\
\leq\;& 2\mathbf{P}\left(\omega(e,e_2) \geq \varepsilon/2\,,\, \min_{i=1,2}\left\{[\omega(e,e_i)\beta(e_i)]^{-1/\mu}\right\} > n\right)\\
\leq\;& 2\mathbf{P}\left(\beta(e_2)^{-1} > n^{\mu}\varepsilon/2\,,\, [\omega(e,e_1)\beta(e_1)]^{-1/\mu} > n\right)\\
\leq\;& 2\mathbf{P}\left(\beta(e_2)^{-1} > n^{\mu}\varepsilon/2\,,\, \omega(e,e_1) \leq n^{-1/2}\right) + 2\mathbf{P}\left(\beta(e_2)^{-1} > n^{\mu}\varepsilon/2\,,\, \beta(e_1)^{-1} > n^{\mu-1/2}\right)\\
=:\;& 2\mathbf{P}(E_1) + 2\mathbf{P}(E_2)\,.
\end{aligned}$$

Let $0<\delta<1$. We have by (7.2) and Lemma 7.1,

$$\begin{aligned}
\mathbf{P}(E_1) &= \mathbf{P}\left(\omega(e,e_1) \leq n^{-1/2}\right)\mathbf{P}\left(\beta(e_2)^{-1} > n^{\mu}\varepsilon/2\right)\\
&\leq c_{32} n^{-1/4} n^{-\delta\mu}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{P}(E_2) &= \mathbf{P}\left(\beta(e_1)^{-1} > n^{\mu-1/2}\right)\mathbf{P}\left(\beta(e_2)^{-1} > n^{\mu}\varepsilon/2\right)\\
&\leq c_{33} n^{-\delta(\mu-1/2)} n^{-\delta\mu}\,.
\end{aligned}$$

It suffices to take $1/4+\delta\mu > 1$ and $\delta(2\mu-1/2) > 1$ to complete the proof, for example by taking $\delta = 4/5$ and $\mu = 19/20$. □

Let $\varepsilon \in (0,1/3)$ be such that

$$\mathbf{E}\left[(\#\{i\,:\,\omega(e_i,e) > 1-\varepsilon\})^{\frac{2-\mu}{1-\mu}}\right] < 1\,. \tag{7.3}$$

Denote by $\mathbb{U}$ the set of the root and all the vertices y such that for any vertex $x \in \mathbb{T}$ with $e < x \leq y$, we have $\omega(x,\overleftarrow{x}) > 1-\varepsilon$; we observe that by (7.3), $\mathbb{U}$ is a subcritical Galton–Watson tree. Denote by U_k the size of the generation k.

Lemma 7.3. *There exists a constant $c_{34} < 1$ such that for any $k \geq 0$*

$$\mathbf{E}\left[U_k^{1/(1-\mu)}\right] \leq c_{34}^k .$$

Proof. By Galton–Watson property,

$$\mathbf{E}\left[U_{k+1}^{1/(1-\mu)}\right] = \mathbf{E}\left[\left(\sum_{i=1}^{U_1} U_k^{(i)}\right)^{1/(1-\mu)}\right]$$

where conditionally on U_1, $U_k^{(i)}$, $i \geq 1$ is a family of i.i.d random variables distributed as U_k. Since $(\sum_{i=1}^n a_i)^p \leq n^p \sum_{i=1}^p a_i^p$ (for $p > 0$ and $a_i \geq 0$), it yields that

$$\begin{aligned}
\mathbf{E}\left[U_{k+1}^{1/(1-\mu)}\right] &\leq \mathbf{E}\left[U_1^{1/(1-\mu)} \sum_{i=1}^{U_1} \left(U_k^{(i)}\right)^{1/(1-\mu)}\right] \\
&= \mathbf{E}\left[U_1^{\frac{2-\mu}{1-\mu}}\right] \mathbf{E}\left[U_k^{1/(1-\mu)}\right] .
\end{aligned}$$

The proof follows from equation (7.3). □

We are now able to complete the proof of Theorem 1.4.

Proof of Theorem 1.4 : the binary tree case. We suppose without loss of generality that $\omega(e, \overleftarrow{e}) \leq 1 - \varepsilon$. For any vertex y, we call Y the youngest ancestor of y such that $\omega(Y, \overleftarrow{Y}) \leq 1 - \varepsilon$. We have for any $n \geq 0$,

$$E_\omega^e[N_n] = \sum_{|y|=n} P_\omega^e(T_y < \infty) E_\omega^y[N(y)] ,$$

where, as before, $N(y) := \sum_{k \geq 0} \mathbb{1}_{\{X_k = y\}}$ and $N_n = \sum_{|y|=n} N(y)$. By the Markov property,

$$E_\omega^y[N(y)] = G(y, Y) + P_\omega^y(T_Y < \infty) P_\omega^Y(T_y < \infty) E_\omega^y[N(y)] ,$$

where $G(y, Y) := E_\omega^y\left[\sum_{k=0}^{T_Y} \mathbb{1}_{\{X_k = y\}}\right]$. It yields that

$$\begin{aligned}
E_\omega^e[N_n] &= \sum_{|y|=n} P_\omega^e(T_y < \infty) \frac{G(y, Y)}{1 - P_\omega^Y(T_y < \infty) P_\omega^y(T_Y < \infty)} \\
&\leq \sum_{|y|=n} P_\omega^e(T_y < \infty) \frac{G(y, Y)}{1 - P_\omega^Y(T_Y^* < \infty)} \\
&\leq \sum_{|y|=n} P_\omega^e(T_y < \infty) \frac{G(y, Y)}{\gamma(Y)} .
\end{aligned}$$

By coupling the walk on $[\![y, Y]\!]$ with a one-dimensional random walk, we see that $P_\omega^y(T_y^* < T_Y) \le \varepsilon + (1-\varepsilon)\frac{\varepsilon}{1-\varepsilon} = 2\varepsilon \le 2/3$, so that $G(y,Y) \le 3$. On the other hand, $P_\omega^e(T_y < \infty) \le P_\omega^e(T_Y < \infty)$. Therefore,

$$\begin{aligned}
\mathbb{E}[N_n] &\le 3\mathbf{E}\left[\sum_{|y|=n} P_\omega^e(T_Y < \infty)\frac{1}{\gamma(Y)}\right] \\
&= 3\mathbf{E}\left[\sum_{|y|=n}\sum_{z=Y} P_\omega^e(T_z < \infty)\frac{1}{\gamma(z)}\right] \\
&= 3\mathbf{E}\left[\sum_{|z|\le n} P_\omega^e(T_z < \infty) \sum_{|y|=n:Y=z}\frac{1}{\gamma(z)}\right].
\end{aligned}$$

By independence and stationarity of the environment,

$$\begin{aligned}
\mathbb{E}[N_n] &\le 3\sum_{|z|\le n}\mathbb{P}(T_z < \infty)\mathbf{E}\left[\sum_{|y|=n-|z|:Y=e}\frac{1}{\gamma(e)}\right] \\
&= 3\sum_{|z|\le n}\mathbb{P}(T_z < \infty)\mathbf{E}\left[\frac{\mathbb{I}_{\{\omega(e,\overleftarrow{e})\le 1-\varepsilon\}}U_{n-|z|}}{\gamma(e)}\right] \\
&\le 3\sum_{|z|\le n}\mathbb{P}(T_z < \infty)\mathbf{E}\left[\left(\frac{\mathbb{I}_{\omega(e,\overleftarrow{e})\le 1-\varepsilon}}{\gamma(e)}\right)^{1/\mu}\right]^{\mu}\mathbf{E}\left[U_{n-|z|}^{1/(1-\mu)}\right]^{1-\mu},
\end{aligned}$$

by the Hölder inequality. We use Lemmas 7.2 and 7.3 to see that

$$\mathbb{E}[N_n] \le c_{35}\sum_{|z|\le n}\mathbb{P}(T(z) < \infty)c_{36}^{n-|z|}.$$

By Lemma 2.1,

$$\mathbb{E}[N_n] \le c_{35}c_1\sum_{k=0}^{n} c_{36}^k < c_{35}c_1/(1-c_{36}).$$

Since $\tau_n \le \sum_{k=-1}^n N_k$ and $N_{-1} \le N_0$, where $\tau_n := \inf\{k \ge 0 : |X_k| = n\}$ as before, we have $\mathbb{E}[\tau_n] \le c_{37}\, n$. Fatou's lemma yields that $\mathbb{P}$-almost surely, $\liminf_{n\to\infty}\frac{\tau_n}{n} < \infty$, which proves that $v > 0$ in view of the relation $\lim_{n\to\infty}\frac{\tau_n}{n} = \frac{1}{v}$. □

8 Proof of Lemmas 5.1 and 3.2

We consider the one-dimensional RWRE $(R_n)_{n\geq 0}$ when we consider the case $\mathbb{T} = \{-1, 0, 1, \ldots\}$. This RWRE is such that the random variables $A(i)$, $i \geq 0$ are independent and have the distribution of A, when we set for $i \geq 0$,

$$A(i) := \frac{\omega(i, i+1)}{\omega(i, i-1)}$$

with $\omega(y, z)$ the quenched probability to jump from y to z. We recall that, as defined in equations (3.3) and (5.1),

$$\begin{aligned} p\,(n, a) &:= \mathbb{P}^0(T_{-1} \wedge T_n > a)\,, \\ m(n, \lambda) &:= \mathbf{E}\left[\left(E^0_\omega\left[T_{-1} \wedge T_n\right]\right)^\lambda\right]\,. \end{aligned}$$

We study the walk $(R_n)_{n\geq 0}$ through its potential. We introduce for $p \geq i \geq 0$, $V(0) = 0$ and

$$\begin{aligned} V(i) &= -\sum_{k=0}^{i-1} \ln(A(k))\,, \\ M(i) &= \max_{0\leq k\leq i} V(k)\,, \\ H_1(i) &= \max_{0\leq k\leq i} V(k) - V(i)\,, \\ H_2(i, p) &= \max_{i\leq k\leq p} V(k) - V(i)\,. \end{aligned}$$

Let us introduce for $t \in \mathbb{R}$ the Laplace transform $\mathbf{E}[A^t]$, and define $\phi(t) := \ln(\mathbf{E}[A^t])$. Denote by I its Legendre transform $I(x) = \sup\{tx - \phi(t), t \in \mathbb{R}\}$ where $x \in \mathbb{R}$. Let also

$$[a, b] := [\text{ess inf}(\ln A), \text{ess sup}(\ln A)]\,.$$

Two situations occur. If $a = b$, it means that A is a constant almost surely. In this case, $I(x) = 0$ if $x = a$ and is infinite otherwise. If $a < b$, then I is finite on $]a, b[$ and infinite on $\mathbb{R}\backslash[a, b]$. Moreover, for any $x \in]a, b[$, we have $I'(x) = t(x)$ where $t(x)$ is the real such that $I(x) = xt(x) - \phi(t(x))$, or, equivalently, $x = \phi'(t(x))$.

We define and compute two useful parameters. Call $\mathcal{D} := \{x_1,\, x_2,\, ,z_1,\, z_2 \in \mathbb{R}^4_+, z_1 + z_2 \leq 1\}$. Define for $\lambda > 0$, and with the convention that $0 \times \infty := 0$,

$$L(\lambda) := \sup_{\mathcal{D}} \left\{ \Big((x_1 z_1) \wedge (x_2 z_2) \Big) \lambda - I(-x_1) z_1 - I(x_2) z_2 \right\}, \tag{8.1}$$

$$L' := \sup \left\{ \frac{x_1 + x_2}{x_1 x_2} \ln(q_1) - \frac{I(-x_1)}{x_1} - \frac{I(x_2)}{x_2}\,,\; x_1, x_2 > 0 \right\}. \tag{8.2}$$

If $q_1 = 0$, we set $L' = -\infty$. Notice that $L(\lambda) \geq 0$ is necessarily reached for $x_1 z_1 = x_2 z_2$. It yields that

$$(8.3) \quad L(\lambda) = 0 \vee \sup \left\{ \frac{x_1 x_2}{x_1 + x_2} \lambda - I(-x_1) \frac{x_2}{x_1 + x_2} - I(x_2) \frac{x_1}{x_1 + x_2} \, , \, x_1, x_2 > 0 \right\},$$

where $c \vee d := \max(c, d)$. The computation of $L(\lambda)$ and L' is done in the following lemma.

Lemma 8.1. *We have*

$$(8.4) \qquad L(\lambda) = 0 \vee \phi(\bar{t}\,),$$

$$(8.5) \qquad L' = -\Lambda,$$

where $\bar{t}$ verifies $\phi(\bar{t}\,) = \phi(\bar{t} + \lambda)$ if it exists and $\bar{t} := 0$ otherwise.

Proof. When A is a constant almost surely, $L(\lambda) = 0$ and (8.4) is true. Therefore we assume that $a < b$. Considering equation (8.3), we see that if $L(\lambda) > 0$, then $L(\lambda)$ is reached by a pair (x_1, x_2) which satisfies :

$$(8.6) \qquad \lambda \frac{x_2}{x_1 + x_2} + \frac{I(-x_1)}{x_1 + x_2} + I'(-x_1) - \frac{I(x_2)}{x_1 + x_2} = 0,$$

$$(8.7) \qquad \lambda \frac{x_1}{x_1 + x_2} - \frac{I(-x_1)}{x_1 + x_2} + \frac{I(x_2)}{x_1 + x_2} - I'(x_2) = 0.$$

We deduce from equations (8.6) and (8.7) that $I'(x_2) - I'(-x_1) = \lambda$, i.e. $t(x_2) - t(-x_1) = \lambda$. Plugging this into (8.3) yields

$$L(\lambda) = 0 \vee \sup \left\{ \frac{\phi(t)\phi'(t + \lambda) - \phi(t + \lambda)\phi'(t)}{\phi'(t + \lambda) - \phi'(t)}, t \in \mathbb{R}, \phi'(t) < 0, \phi'(t + \lambda) > 0 \right\}.$$

Let $h(t) := \frac{\phi(t)\phi'(t+\lambda) - \phi(t+\lambda)\phi'(t)}{\phi'(t+\lambda) - \phi'(t)}$. Then $L(\lambda) = 0 \vee h(\bar{t}\,)$ where $\bar{t}$ verifies $h'(\bar{t}\,) = 0$, which is equivalent to say that $\phi(\bar{t}\,) = \phi(\bar{t} + \lambda)$. We find that $h(\bar{t}\,) = \phi(\bar{t}\,)$, which gives (8.4). The computation of (8.5) is similar and is therefore omitted. □

8.1 Proof of Lemma 5.1

We begin by some notation. Let $A > 0$ and $B > 0$ be two expressions which can depend on any variable, and in particular on n. We say that $A \lesssim B$ if we can find a function f of the variable n such that $\lim_{n\to\infty}\frac{1}{n}\ln(f(n))=0$ and $A\le f(n)B$. We say that $A\simeq B$ if $A\lesssim B$ and $B\lesssim A$. By circuit analogy (see [DS84]), we find for $0\le i\le n$,

$$P_\omega^0\left(T_i<T_{-1}\right)=\frac{1}{e^{V(0)}+e^{V(1)}+\ldots+e^{V(i)}}\,.$$

It follows that

$$\frac{e^{-M(i)}}{n+1}\le P_\omega^0\left(T_i<T_{-1}\right)\le e^{-M(i)}\,. \tag{8.8}$$

We deduce also that

$$\frac{e^{-H_2(i,n)}}{n+1}\le P_\omega^{i+1}\left(T_n<T_i\right)\le e^{-H_2(i,n)}\,, \tag{8.9}$$

$$\frac{e^{-H_1(i)}}{n+1}\le P_\omega^{i-1}\left(T_{-1}<T_i\right)\le e^{-H_1(i)}\,. \tag{8.10}$$

Finally, the quenched expectation $G\left(i,-1\wedge n\right)$ of the number of times the walk starting from i returns to i before reaching -1 or n verifies

$$G\left(i,-1\wedge n\right)\;=\;\left\{\omega(i,i-1)P_\omega^{i-1}\left(T_{-1}<T_i\right)+\omega(i,i+1)P_\omega^{i+1}\left(T_n<T_i\right)\right\}^{-1}\,,$$

so that

$$c_{37}e^{H_1(i)\wedge H_2(i,n)}\le G(i,-1\wedge n)\le c_{38}(n+1)e^{H_1(i)\wedge H_2(i,n)}\,.$$

Since $E_\omega^0[T_{-1}\wedge T_n]=1+\sum_{i=0}^{n-1}P_\omega^0\left(T_i<T_{-1}\right)G\left(i,-1\wedge n\right)$, we get

$$1+\frac{c_{37}}{n+1}\max_{0\le i\le n}e^{-M(i)+H_1(i)\wedge H_2(i,n)}\le E_\omega^0[T_{-1}\wedge T_n] \tag{8.11}$$

and

$$E_\omega^0[T_{-1}\wedge T_n]\le 1+c_{38}n(n+1)\max_{0\le i\le n}e^{-M(i)+H_1(i)\wedge H_2(i,n)}\,.$$

As a result,

$$\mathbf{E}\big[\big(E_\omega^0[T_{-1}\wedge T_n]\big)^\lambda\big]\simeq\max_{0\le i\le n}\mathbf{E}\left[e^{\lambda[-M(i)+H_1(i)\wedge H_2(i,n)]}\right]\,. \tag{8.12}$$

We proceed to the proof of Lemma 5.1. Let $\eta>0$ and $0\le i\le n$. Let $\varepsilon>0$ be such that $(|a|\vee|b|)\varepsilon<\eta$. For fixed i and n, we denote by K_1 and K_2 the integers such that

$$\begin{aligned}K_1\eta\le\quad &H_1(i)\quad<(K_1+1)\eta\,,\\ K_2\eta\le\quad &H_2(i,n)\quad<(K_2+1)\eta\,.\end{aligned}$$

Similarly, let L_1 and L_2 be integers such that

$$\begin{aligned} &\exists \;\; L_1\lfloor \varepsilon n\rfloor \le x < (L_1+1)\lfloor \varepsilon n\rfloor \quad \text{such that} \quad H_1(i) = V(i-x) - V(i)\,, \\ &\exists \;\; L_2\lfloor \varepsilon n\rfloor \le y < (L_2+1)\lfloor \varepsilon n\rfloor \quad \text{such that} \quad H_2(i,n) = V(i+y) - V(i)\,. \end{aligned}$$

Finally, $e^{\lambda[-M(i)+H_1(i)\wedge H_2(i,n)]} \le e^{(K_1\wedge K_2+1)\lambda\eta n}$. By our choice of ε, we have for any integers k_1, k_2, ℓ_1, ℓ_2,

$$\begin{aligned} \mathbb{P}\left(K_1 = k_1,\, L_1 = \ell_1\right) &\le \mathbb{P}\Big(V(\ell_1\lfloor \varepsilon n\rfloor) \in [-(k_1+2)\eta n, -(k_1-1)\eta n]\Big)\,, \\ \mathbb{P}\left(K_2 = k_2,\, L_2 = \ell_2\right) &\le \mathbb{P}\Big(V\left(\ell_2\lfloor \varepsilon n\rfloor\right) \in [(k_2-1)\eta n, (k_2+2)\eta n]\Big)\,. \end{aligned}$$

By Cramér's theorem (see [dH00] for example),

$$\begin{aligned} \mathbb{P}\Big(V(\ell_1\lfloor \varepsilon n\rfloor) \in [-(k_1+2)\eta n, -(k_1-1)\eta n]\Big) &\lesssim \exp\Big(-\ell_1\lfloor \varepsilon n\rfloor(I(-x_1) - \lambda\eta)\Big) \\ \mathbb{P}\Big(V(\ell_2\lfloor \varepsilon n\rfloor) \in [(k_2-1)\eta n, (k_2+2)\eta n]\Big) &\lesssim \exp\Big(-\ell_2\lfloor \varepsilon n\rfloor(I(x_2) - \lambda\eta)\Big) \end{aligned}$$

if $-x_1$ is the point of $\left[\frac{-(k_1+2)\eta n}{\ell_1\lfloor \varepsilon n\rfloor}, \frac{-(k_1-1)\eta n}{\ell_1\lfloor \varepsilon n\rfloor}\right]$ where I reaches the minimum on this interval, and x_2 is the equivalent in $\left[\frac{(k_2-1)\eta n}{\ell_2\lfloor \varepsilon n\rfloor}, \frac{(k_2+2)\eta n}{\ell_2\lfloor \varepsilon n\rfloor}\right]$. It yields that

$$\begin{aligned} &\mathbb{E}\left[e^{\lambda[-M(i)+H_1(i)\wedge H_2(i,n)]}\right] \\ \lesssim\; &\max_{k_1,k_2,\ell_1,\ell_2\in D'} \exp\left(\left(k_1\wedge k_2\right)\lambda\eta n - I(-x_1)\ell_1\lfloor \varepsilon n\rfloor - I(x_2)\ell_2\lfloor \varepsilon n\rfloor + 3\lambda\eta n\right)\,, \end{aligned}$$

where D' is the (finite) set of all possible values of (K_1, K_2, L_1, L_2). We note that

$$\begin{aligned} &(k_1\wedge k_2)\lambda\eta n - I(-x_1)\ell_1\lfloor \varepsilon n\rfloor - I(x_2)\ell_2\lfloor \varepsilon n\rfloor \\ \le\; &(x_1\ell_1\lfloor \varepsilon n\rfloor \wedge x_2\ell_2\lfloor \varepsilon n\rfloor)\lambda - I(-x_1)\ell_1\lfloor \varepsilon n\rfloor - I(x_2)\ell_2\lfloor \varepsilon n\rfloor + 3\lambda\eta n \\ \le\; &(L(\lambda) + 3\lambda\eta)n \end{aligned}$$

by (8.1). Finally, $\mathbb{E}[e^{\lambda(-M(i)+H_1(i)\wedge H_2(i,n))}] \lesssim e^{n(L(\lambda)+6\lambda\eta)}$ so that, by equation (8.12), $m(n,\lambda) \lesssim e^{n(L(\lambda)+6\lambda\eta)}$. We let η tend to 0 to get that

$$\limsup_{n\to\infty} \frac{1}{n}\ln(m(n,\lambda)) \le L(\lambda)\,.$$

Let $\lambda < \Lambda$. By definition of Λ and equation (8.4), it implies that $L(\lambda) < \frac{1}{q_1}$, so that we can find $r > q_1$ such that $\sum_{n\ge 0} m(n,\lambda) r^n < \infty$. It means that $\lambda \le \lambda_c$. Consequently, $\Lambda \le \lambda_c$. $\square$

8.2 Proof of Lemma 3.2

Fix x_1, $x_2 > 0$. Write

$$z_1 = \frac{x_2}{x_1+x_2}, \quad z_2 = \frac{x_1}{x_1+x_2}, \quad z = \frac{x_1 x_2}{x_1+x_2}\,.$$

Let $a \geq 100$ and $n = n(a) := \lfloor \frac{\ln(a)}{z} \rfloor$. We have, by the strong Markov property, $P_\omega^0(T_{-1}\wedge T_n > a) \geq P_\omega^0(T_{\lfloor z_1 n\rfloor} < T_{-1}) P_\omega^{\lfloor z_1 n\rfloor}(T_{\lfloor z_1 n\rfloor} < T_{-1}\wedge T_n)^a$. It follows by (8.8), (8.9) and (8.10) that

$$\begin{aligned}
p\,(n,a) &\gtrsim \mathbf{E}\Big[e^{-M(\lfloor z_1 n\rfloor)}\Big(1-e^{-H_1(\lfloor z_1 n\rfloor)\wedge H_2(\lfloor z_1 n\rfloor, n)}\Big)^a\Big] \\
&\geq (1-e^{-zn})^a \mathbf{P}\Big(V\,(\lfloor z_1 n\rfloor) < -zn,\, M\,(\lfloor z_1 n\rfloor) \leq 0\Big)\mathbf{P}\Big(V\,(\lfloor z_2 n\rfloor +1) > zn\Big) \\
&\gtrsim \mathbf{P}\Big(V\,(\lfloor z_1 n\rfloor) < -zn,\, M\,(\lfloor z_1 n\rfloor) \leq 0\Big)\mathbf{P}\Big(V\,(\lfloor z_2 n\rfloor +1) > zn\Big)
\end{aligned}$$

by our choice of n. Let $k \geq 0$. Call τ the first time when the walk $(V(i))_{i\geq 0}$ reaches its maximum on $[0,k]$. Let $i \in [0,k]$ and for $0 \leq r \leq k-1$, $X_r := \ln(A_{\bar r})$ where $\bar r := i + r$ modulo k. We observe that

$$\begin{aligned}
\mathbf{P}(V_k < -zn,\, \tau = i) &\leq \mathbf{P}(X_0 + \ldots + X_{k-1} < -zn,\, X_0 + \ldots + X_j \leq 0 \ \ \forall\ \ 0 \leq j \leq k-1) \\
&= \mathbf{P}(V_k < -zn,\, M_k \leq 0)\,.
\end{aligned}$$

We obtain that $\mathbf{P}\,(V_k < -zn, M_k \leq 0) \geq \frac{1}{k+1}\mathbf{P}\,(V_k < -zn)$. Therefore, for any $\varepsilon > 0$,

$$\begin{aligned}
p\,(n,a) &\gtrsim \mathbf{P}\Big(V\,(\lfloor z_1 n\rfloor) < -zn\Big)\mathbf{P}\Big(V\,(\lfloor z_2 n\rfloor +1) > zn\Big) \\
&\gtrsim \exp\Big(n\,(-I(-x_1)z_1 - I(x_2)z_2 - 2\varepsilon)\Big)
\end{aligned}$$

by Cramér's theorem. It yields that

$$\begin{aligned}
\liminf_{a\to\infty}\left\{\sup_{\ell\geq 0}\frac{\ln(q_1^\ell p\,(\ell,a))}{\ln(a)}\right\} &\geq \liminf_{a\to\infty}\frac{\ln(q_1^n p\,(n,a))}{\ln(a)} \\
&\geq \frac{\ln(q_1) - I(-x_1)z_1 - I(x_2)z_2 - 2\varepsilon}{z}\,.
\end{aligned}$$

Finally, by (8.2) and (8.5),

$$\liminf_{a\to\infty}\left\{\sup_{n\geq 0}\frac{\ln(q_1^n p(n,a))}{\ln(a)}\right\} \geq L' = -\Lambda\,. \ \square$$

9 The critical case $\Lambda = 1$

We complete Theorem 1.1 with the following

Proposition 9.1. *When $\Lambda = 1$, the transient RWRE on a Galton–Watson tree has zero speed.*

Proof of Proposition 9.1. By (iv) of Fact, we know that the speed v verifies $v = \frac{E_{\mathbb{S}}[X_{\Gamma_1}]}{E_{\mathbb{S}}[\Gamma_1]}$. Therefore, it is enough to show that $E_{\mathbb{S}}[\Gamma_1] = \infty$. With the reasoning (and the notation) of Lemma 3.3, there exists a constant d_1 such that

$$E_{\mathbb{S}}[\Gamma_1] \geq d_1 E_{\mathbb{Q}}[T_{\overleftarrow{e}} \wedge T_{h(e)}]$$

where $h(e)$ is the vertex of minimal generation such that $\nu(h(e)) > 1$ ($h(e) = e$ if $\nu(e) > 1$). We observe that $|h(e)|$ is a geometric variable of parameter q_1. Since we already know that $E_{\mathbb{Q}}[T_{\overleftarrow{e}} \wedge T_{h(e)}]$ is the expectation of $T_{-1} \wedge T_{|h(e)|}$ for the one-dimensional RWRE $(R_n, n \geq 0)$ associated with A (cf Section 3.2), we only have to prove that if G is a geometric random variable with parameter q_1 and $\Lambda = 1$ (in fact also $\Lambda \leq 1$), the one-dimensional RWRE (R_n) is such that

$$\mathbb{E}[T_{-1} \wedge T_G] = \infty\,, \tag{9.1}$$

where $\mathbb{E}$ denotes as before the annealed probability for the RWRE R_n. We will denote by P_ω the quenched probability, by $\mathbf{P}$ the distribution of the environment. We begin with a lemma.

Lemma 9.2. *Let S_n be a random walk with $P(S_1 > 0) > 0$ and $S_0 = 0$, G a geometric random variable with parameter q_1, and we suppose that there is a non-negative real κ such that $E[e^{\kappa S_1}] = \frac{1}{q_1}$, and $E[S_1^+ e^{\kappa S_1}] < \infty$. We define for $x \geq 0$*

$$\tau_x := \inf\{k \geq 0 \,:\, S_k \geq x\}\,.$$

Then,

- *If the distribution of S_1 is non-lattice, $e^{\kappa x} P(\tau_x < G)$ has a positive limit when $x \to \infty$.*
- *If the distribution of S_1 is lattice with maximal span h, $e^{\kappa h n} P(\tau_{nh} < G)$ converges to a positive constant when $n \to \infty$.*

Proof of Lemma 9.2. We make use of the renewal theory. We denote by $\widetilde{S}$ the random walk S killed at time G ($\widetilde{S}_k = S_k$ for $0 \leq k \leq G$). Let H_1, $H_2, \ldots$ denote the sequence of strict ascending ladder heights of S and L_1, $L_2, \dots$ the corresponding ladder epochs (possibly infinite). These variables are defined by

$$\begin{aligned} L_{i+1} &:= \inf\{k > L_i \,:\, S_k > H_i\}\,, \\ H_{i+1} &:= S_{L_{i+1}} \qquad \text{if } L_{i+1} < \infty\,, \end{aligned}$$

where we set $L_0 = H_0 = 0$ and $\inf \emptyset = \infty$. We use with natural notation $\widetilde{H_1}$, $\widetilde{H}_2, \dots$ and $\widetilde{L}_1$, $\widetilde{L}_2, \dots$ the equivalent for $\widetilde{S}$. We observe that the maximum $\widetilde{M}$ of $\widetilde{S}$ verifies

$$\widetilde{M} := \max\{\widetilde{H_1} + \ldots + \widetilde{H}_k, k \geq 0 \text{ such that } \widetilde{L}_k < \infty\}\,.$$

Finally, let $\widetilde{\kappa}$ be such that

$$E[e^{\widetilde{\kappa}\widetilde{H}_1} \mathbb{I}_{\{\widetilde{L}_1 < \infty\}}] = 1\,.$$

We show that $\widetilde{\kappa}$ exists and is equal to κ. Let $Y_n := e^{\kappa S_n} \mathbb{I}_{\{G \geq n\}}$. By the definition of κ , the process $(Y_n)_{n\geq 0}$ is a nonnegative martingale. Therefore, since $\widetilde{L}_1$ is a stopping time, $(Y_{n\wedge \widetilde{L}_1})_{n\geq 0}$ is a nonnegative martingale. By Fatou's Lemma,

$$E[Y_{\widetilde{L}_1}] \leq E[Y_0] = 1\,.$$

We observe that for any n, and since κ is supposed nonnegative, $Y_{n\wedge \widetilde{L}_1} \leq \max\{1, Y_{\widetilde{L}_1}\}$. Therefore, $(Y_{n\wedge \widetilde{L}_1})_{n\geq 0}$ is a uniformly integrable martingale, which yields that

$$E[Y_{\widetilde{L}_1}] = 1\,.$$

Besides, $Y_{\widetilde{L}_1} = e^{\kappa \widetilde{H}_1} \mathbb{I}_{\{\widetilde{L}_1 < \infty\}}$. We deduce that

$$E[e^{\kappa\widetilde{H}_1} \mathbb{I}_{\{\widetilde{L}_1 < \infty\}}] = 1\,.$$

Hence, $\widetilde{\kappa} = \kappa$. Let us deal with the non-lattice case first. By Chapter XI Section 6 of [Fel71], under our assumptions, we know that for some constant $d_2 > 0$,

$$P(\widetilde{M} \geq x) \sim d_2 e^{-\widetilde{\kappa} x} \qquad x \to \infty\,.$$

Since $P(\tau_x < G) = P(\widetilde{M} \geq x)$, and $\widetilde{\kappa} = \kappa$, we obtain the lemma in this case. We turn now to the lattice case. We prove the following equation.

$$P(\widetilde{M} \geq hn) \sim d_3 e^{-\kappa h n} \qquad n \to \infty\,,$$

where h is the maximal span of the lattice distribution. We notice that

$$P(\widetilde{M} \geq hn) = P(\widetilde{H}_1 \geq hn, \widetilde{L}_1 < \infty) + \sum_{k=1}^{n-1} P(\widetilde{H}_1 = hk) P(\widetilde{M} \geq h(n-k)) .$$

We can write it as

$$u_n = b_n + (a_1 u_{n-1} + a_1 u_{n-1} + \ldots + a_{n-1} u_1)$$

with

$$\begin{aligned} u_n &:= P(\widetilde{M} \geq hn) , \\ a_n &:= P(\widetilde{H}_1 = hn) , \\ b_n &:= P(\widetilde{H}_1 \geq hn, \widetilde{L}_1 < \infty) . \end{aligned}$$

We have proved that $\sum_{n\geq 0} a_n e^{\kappa hn} = 1$. We apply Theorem 1 of Section XIII.4 in [Fel68] to the sequence $(e^{\kappa hn} u_n, e^{\kappa hn} a_n, e^{\kappa hn} b_n)$, to obtain that $e^{\kappa hn} u_n$ converges to a positive constant. The lemma follows. $\square$

We go back to the proof of equation (9.1). Let G be a geometric random variable with parameter q_1. With the notation of Section 8, we have by equation (8.11)

$$\begin{aligned} \mathbb{E}[T_{-1} \wedge T_G] &\geq 1 + c_{37} \mathbf{E} \left[\frac{1}{1+G} \max_{0 \leq i \leq G} \{ e^{-M(i) + H_1(i) \wedge H_2(i,G)} \} \right] . \\ &=: 1 + c_{37} \mathbf{E}[Z] . \end{aligned} \tag{9.2}$$

We are reduced to the study of the potential V. For sake of concision, we focus on the situation where V_1 has a non-lattice distribution. The lattice case presents no further difficulty. We denote by τ_x the first passage time of the level x by the walk $(V_n,\, n \geq 0)$. By the definition of M, H_1 and H_2, we have for any positive reals δ and z,

$$\mathbf{P}\left(Z > \frac{1}{1 + 2\delta x} e^{x-z} \right) \geq \mathbf{P}(\tau_z > G,\, \tau_{-x} < G,\, G < \delta x) \mathbf{P}(\tau_x < G,\, G < \delta x) . \tag{9.3}$$

First, we observe that

$$\begin{aligned} \mathbf{P}(\tau_z > G,\, \tau_{-x} < G,\, G < \delta x) &\geq \mathbf{P}(\tau_z > G,\, \tau_{-x} < G) - \mathbf{P}(G \geq \delta x) \\ &= \mathbf{P}(\tau_z > G,\, \tau_{-x} < G) - q_1^{\delta x} . \end{aligned}$$

We see that

$$\mathbf{P}(\tau_z > G,\, \tau_{-x} < G) = \mathbf{P}(\tau_{-x} < G) - \mathbf{P}(\tau_z < \tau_{-x} < G) - \mathbf{P}(\tau_{-x} < \tau_z < G) . \tag{9.4}$$

By the Markov property and the lack of memory of the geometric distribution, we have

$$\mathbf{P}(\tau_z < \tau_{-x} < G) = \mathbf{P}(\tau_z < G)\mathbf{P}(\tau_{-x-z} < G) \le \mathbf{P}(\tau_z < G)\mathbf{P}(\tau_{-x} < G)$$

and

$$\mathbf{P}(\tau_{-x} < \tau_z < G) = \mathbf{P}(\tau_{-x} < G)\mathbf{P}(\tau_{x+z} < G) \le \mathbf{P}(\tau_{-x} < G)\mathbf{P}(\tau_z < G)\,.$$

By equation (9.4), we get

$$\mathbf{P}(\tau_z > G,\, \tau_{-x} < G) \ge \mathbf{P}(\tau_{-x} < G)(1 - 2\mathbf{P}(\tau_z < G))\,.$$

By Lemma 9.2 applied to the random walk $-V$, we know that

$$\mathbf{P}(\tau_{-x} < G) \sim d_4 e^{-\kappa_1 x} \qquad x \to \infty\,,$$

where $\kappa_1 \ge 0$ verifies $\frac{1}{q_1} = \mathbf{E}[e^{-\kappa_1 V_1}] = \mathbf{E}[A^{\kappa_1}]$. We choose $z > 0$ such that $d_5 := d_4(1 - 2\mathbf{P}(\tau_z < G))/2 > 0$. Consequently, for x large enough,

$$\mathbf{P}(\tau_z > G,\, \tau_{-x} < G, G < \delta x) \ge d_5 e^{-\kappa_1 x} - q_1^{\delta x}\,.$$

Similarly,

$$\mathbf{P}(\tau_x < G,\, G < \delta x) \ge \mathbf{P}(\tau_x < G) - q_1^{\delta x} \ge d_6 e^{-\kappa_2 x} - q_1^{\delta x}\,,$$

where κ_2 is the positive real such that $\mathbf{E}[A^{-\kappa_2}] = \frac{1}{q_1}$. By choosing δ large enough, equation (9.3) implies for sufficiently large x,

$$\mathbf{P}(Z > \frac{1}{1+2\delta x} e^{x-z}) \ge d_7 e^{-(\kappa_1+\kappa_2)x} = d_7 e^{-\Lambda x} = d_7 e^{-x}\,.$$

Therefore, if $y = x + \ln(\delta x) + z$, we obtain

$$\mathbf{P}(Z > \frac{1}{1+2\delta y} e^{y-z}) \ge d_7 e^{-y}$$

which yields

$$\mathbf{P}(Z > e^x) \ge d_7 e^{-z} \frac{e^{-x}}{x}\,.$$

This implies that for x large enough, $\mathbf{P}(Z > x) \ge \frac{d_8}{x\ln(x)}$. Therefore :

$$E[Z] = \int_0^\infty \mathbf{P}(Z > x)dx = \infty\,.$$

By (9.2), we get $\mathbb{E}[T_{-1} \wedge T_G] = \infty$, which completes the proof of equation (9.1). $\square$

Acknowledgements : I would like to thank Zhan Shi for suggesting me the problem and for many precious discussions. I would also like to thank the referees for their helpful comments. Finally I thank Alexander Fribergh and Christophe Sabot who kindly suggested me the proof of the critical case.

Chapitre III

Large deviations for transient random walks in random environment on a Galton–Watson tree [1]

Summary. Consider a random walk in random environment on a supercritical Galton–Watson tree, and let τ_n be the hitting time of generation n. The paper presents a large deviation principle for τ_n/n, both in quenched and annealed cases. Then we investigate the subexponential situation, revealing a polynomial regime similar to the one encountered in one dimension. The paper heavily relies on estimates on the tail distribution of the first regeneration time.
Key words. Random walk in random environment, law of large numbers, large deviations, Galton–Watson tree.

AMS subject classifications. 60K37, 60J80, 60F15, 60F10.

1 Introduction

We consider a super-critical Galton–Watson tree $\mathbb{T}$ of root e and offspring distribution $(q_k,\, k \geq 0)$ with finite mean $m := \sum_{k\geq 0} kq_k > 1$. For any vertex x of $\mathbb{T}$, we call $|x|$ the generation of x, ($|e| = 0$) and $\nu(x)$ the number of children of x ; we denote these children by x_i, $1 \leq i \leq \nu(x)$. We let ν_{min} be the minimal integer such that $q_{\nu_{min}} > 0$ and we suppose that $\nu_{min} \geq 1$ (thus $q_0 = 0$). In particular, the tree survives almost surely. Following Pemantle

1. This chapter comes from an article to appear in *Annales de l'Institut Henri Poincaré*. The last section, which does not belong to the original paper, deals with the bound on the speed of the RWRE.

and Peres [PP95], on each vertex x, we pick independently and with the same distribution a random variable $A(x)$, and we define

- $\omega(x, x_i) := \frac{A(x_i)}{1+\sum_{i=1}^{\nu(x)} A(x_i)}$, $\forall\, 1 \le i \le \nu(x)$,
- $\omega(x, \overleftarrow{x}) := \frac{1}{1+\sum_{i=1}^{\nu(x)} A(x_i)}$.

To deal with the case $x = e$, we add a parent $\overleftarrow{e}$ to the root and we set $\omega(\overleftarrow{e}, e) = 1$. Once the environment built, we define the random walk $(X_n,\, n \ge 0)$ starting from $y \in \mathbb{T}$ by

$$\begin{aligned} P_\omega^y(X_0 = y) &= 1, \\ P_\omega^y(X_{n+1} = z \,|\, X_n = x) &= \omega(x, z)\,. \end{aligned}$$

The walk $(X_n,\, n \ge 0)$ is a $\mathbb{T}$-valued Random Walk in Random Environment (RWRE). To determine the transience or recurrence of the random walk, Lyons and Pemantle [LP92] provides us with the following criterion. Let A be a generic random variable having the distribution of $A(e)$.

Theorem A (Lyons and Pemantle [LP92]) *The walk (X_n) is transient if $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, and is recurrent otherwise.*

In the transient case, let v denote the speed of the walk, which is the deterministic real $v \ge 0$ such that

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v, \qquad a.s.$$

Define

$$\begin{aligned} i &:= \text{ess inf}\, A\,, \\ s &:= \text{ess sup}\, A\,. \end{aligned}$$

We make the hypothesis that $0 < i \le s < \infty$. Under this assumption, we gave a criterion in chapter II of the present work for the positivity of the speed v. Let

$$\Lambda := Leb\left\{t \in \mathbb{R} : \mathbf{E}[A^t] \le \frac{1}{q_1}\right\} \qquad (\Lambda = \infty \text{ if } q_1 = 0). \tag{1.1}$$

Theorem B (Chapter II) *Assume $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, and let Λ be as in (1.1).*

(a) *If $\Lambda \le 1$, the walk has zero speed.*

(b) *If $\Lambda > 1$, the walk has positive speed.*

When the speed is positive, we would like to have information on how hard it is for the walk to have atypical behaviours, which means to go a little faster or slower than its natural pace. Such questions have been discussed in the setting of biased random walks on Galton–Watson trees, by Dembo et al. in [DGPZ02]. The authors exhibit a large deviation principle both in quenched and annealed cases. Besides, an uncertainty principle allows them to obtain the equality of the two rate functions. For the RWRE in dimensions one or more, we refer to Zeitouni [Zei04] for a review of the subject. In our case, we consider a random walk which always avoids the parent $\overleftarrow{e}$ of the root, and we obtain a large deviation principle, which, following [DGPZ02], has been divided into two parts.

We suppose in the rest of the paper that

$$\inf_{[0,1]} \mathbf{E}[A^t] \;>\; \frac{1}{m}\,, \tag{1.2}$$

$$\Lambda \;>\; 1\,, \tag{1.3}$$

which ensures that the walk is transient with positive speed. Before the statement of the results, let us introduce some notation. Define for any $n \geq 0$ and $x \in \mathbb{T}$,

$$\begin{aligned} \tau_n \;&:=\; \inf\left\{k \geq 0\, : |X_k| = n\right\}\,, \\ D(x) \;&:=\; \inf\left\{k \geq 1\, : X_{k-1} = x,\, X_k = \overleftarrow{x}\right\},\quad (\inf \emptyset := \infty)\,. \end{aligned}$$

Let $\mathbf{P}$ denote the distribution of the environment ω conditionally on $\mathbb{T}$, and $\mathbf{Q} := \int \mathbf{P}(\cdot)GW(\mathrm{d}\mathbb{T})$. Similarly, we denote by $\mathbb{P}^x$ the distribution defined by $\mathbb{P}^x(\cdot) := \int P^x_\omega(\cdot)\mathbf{P}(\mathrm{d}\omega)$ and by $\mathbb{Q}^x$ the distribution

$$\mathbb{Q}^x(\cdot) := \int \mathbb{P}^x(\cdot)GW(\mathrm{d}\mathbb{T})\,.$$

Theorem 1.1. (Speed-up case) *There exist two continuous, convex and strictly decreasing functions $I_a \leq I_q$ from $[1, 1/v]$ to $\mathbb{R}_+$, such that $I_a(1/v) = I_q(1/v) = 0$ and for $a < b$, $b \in [1, 1/v]$, and almost every ω,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e\left(\frac{\tau_n}{n} \in]a, b]\right) \;=\; -I_a(b)\,, \tag{1.4}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln P^e_\omega\left(\frac{\tau_n}{n} \in]a, b]\right) \;=\; -I_q(b)\,. \tag{1.5}$$

Theorem 1.2. (Slowdown case) *There exist two continuous, convex functions $I_a \leq I_q$ from $[1/v, +\infty[$ to $\mathbb{R}_+$, such that $I_a(1/v) = I_q(1/v) = 0$ and for any $1/v \leq a < b$, and almost* *every ω,*

$$\lim_{n\to\infty}\frac{1}{n}\ln \mathbb{Q}^e\left(\frac{\tau_n}{n}\in[a,b[\right) = -I_a(a)\,, \tag{1.6}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln P^e_\omega\left(\frac{\tau_n}{n}\in[a,b[\right) = -I_q(a)\,. \tag{1.7}$$

Besides, if $i>\nu_{min}^{-1}$, then I_a and I_q are strictly increasing on $[1/v,+\infty[$. When $i\le\nu_{min}^{-1}$, we have $I_a=I_q=0$ on the interval.

As pointed by an anonymous referee, it would be interesting to know when I_a and I_q coincide. We do not know the answer in general. However, the computation of the value of the rate functions at $b=1$ reveals situations where the rate functions differ. Let

$$\psi(\theta) := \ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)}\omega(e,e_i)^\theta\right]\right).$$

Then $\psi(0)=\ln(m)$ and $\psi(1)=\ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)}\omega(e,e_i)\right]\right)$.

Proposition 1.3. *We have*

$$I_a(1) = -\psi\,(1)\;, \tag{1.8}$$

$$I_q(1) = -\inf_{]0,1]}\frac{1}{\theta}\psi(\theta)\,. \tag{1.9}$$

In particular, $I_a(1)=I_q(1)$ if and only if $\psi'(1)\le\psi(1)$. Otherwise $I_a(1)<I_q(1)$.

Quite surprisingly, we can exhibit elliptic environments on a regular tree for which the rate functions differ. This could hint that the uncertainty of the location of the first passage in [DGPZ02] does not hold anymore for a random environment. Here is an explicit example. Consider a binary tree ($q_2=1$). Let A equal 0.01 with probability 0.8 and 500 with probability 0.2. Then we check that the walk is transient, but $\psi'(1)>\psi(1)$ so that $I_a(1)\neq I_q(1)$ on such an environment.

Theorem 1.2 exhibits a subexponential regime in the slowdown case when $i\le\nu_{min}^{-1}$. The following theorem details this regime. Let

$$\mathbb{S}^e(\cdot) \;:=\; \mathbb{Q}^e(.\,|\,D(e)=\infty)\,.$$

Theorem 1.4. *We place ourself in the case $i < \nu_{min}^{-1}$.*
(i) Suppose that either "$i < \nu_{min}^{-1}$ and $q_1 = 0$" or "$i < \nu_{min}^{-1}$ and $s < 1$". There exist constants $d_1, d_2 \in (0,1)$ such that for any $a > 1/v$ and n large enough,

$$e^{-n^{d_1}} < \mathbb{S}^e(\tau_n > an) < e^{-n^{d_2}} . \tag{1.10}$$

(ii) If $q_1 > 0$ and $s > 1$ (id est when $\Lambda < \infty$), the regime is polynomial and we have for any $a > 1/v$,

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \ln\left(\mathbb{S}^e(\tau_n > an)\right) = 1 - \Lambda . \tag{1.11}$$

We mention that in one dimension, which can be seen as a critical state of our model where $q_1 = 1$, such a polynomial regime is proved by Dembo et al. [DPZ96], our parameter Λ taking the place of the well-known κ of Kesten, Kozlov, Spitzer [KKS75]. We did not deal with the critical case $i = \nu_{min}^{-1}$. Furthermore, we do not have any conjecture on the optimal values of d_1 and d_2 and do not know if the two values are equal.

The last part of this chapter deals with a bound on the speed v of the RWRE. We prove that

Proposition 1.5. *The speed v verifies*

$$\frac{E_{\mathbb{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right] - 1}{E_{\mathbb{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right] + 1} \geq v \geq \frac{1 - E_{\mathbb{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right]}{1 + E_{\mathbb{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right]}$$

When considering A deterministic, the RWRE boils down to a biased random walk on a Galton–Watson tree. In this setting, Virag [Vir00] for the upper bound and Chen [Che97] for the lower bound have already proved these inequalities.

The rest of the paper is organized as follows. Section 2 describes the tail distribution of the first regeneration time, which is a preparatory step for the proof of the different theorems. Then we prove Theorems 1.1 and 1.2 in Section 3, which includes also the computation of the rate functions at speed 1 presented in Proposition 1.3. Section 4 is devoted to the subexponential regime with the proof of Theorem 1.4. Finally, Section 5 proves Proposition 1.5.

2 Moments of the first regeneration time

We define the first regeneration time

$$\Gamma_1 := \inf\left\{k > 0 \,:\, \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\right\}$$

as the first time when the walk reaches a generation by a vertex having more than two children and never returns to its parent. We propose in this section to give information on the tail distribution of Γ_1 under $\mathbb{S}^e$. We first introduce some notation used throughout the paper. For any $x \in \mathbb{T}$, let

$$\begin{aligned} N(x) &:= \sum_{k\geq 0} \mathbb{I}_{\{X_k = x\}}\,, \\ T_x &:= \inf\{k \geq 0 :\ X_k = x\}\,, \\ T_x^* &:= \inf\{k \geq 1 \,:\, X_k = x\}\,. \end{aligned} \tag{2.1}$$

This permits to define

$$\begin{aligned} \beta(x) &:= P_\omega^x(T_{\overleftarrow{x}} = \infty)\,, \\ \gamma(x) &:= P_\omega^x(T_{\overleftarrow{x}} = T_x^* = \infty)\,. \end{aligned} \tag{2.2}$$

The following fact can be found in [DGPZ02] (Lemma 4.2) in the case of biased random walks, and is directly adaptable in our setting.

Fact A The first regeneration height $|X_{\Gamma_1}|$ admits exponential moments under the measure $\mathbb{S}^e(\cdot)$.

2.1 The case $i > \nu_{min}^{-1}$

This section is devoted to the case $i > \nu_{min}^{-1}$, where Γ_1 is proved to have exponential moments.

Proposition 2.1. *Suppose that $i > \nu_{min}^{-1}$. There exists $\theta > 0$ such that $E_{\mathbb{S}^e}\left[e^{\theta\Gamma_1}\right] < \infty$.*

Proof. The proof follows the strategy of Proposition 1 of Piau [Pia98]. We couple the distance of our RWRE to the root $(|X_n|)_{n\geq 0}$ with a biased random walk $(Y_n)_{n\geq 0}$ on $\mathbb{Z}$ as follows. Let

$p := \frac{i\nu_{min}}{1+i\nu_{min}}$, and let u_n, $n \geq 0$, be a family of i.i.d. uniformly distributed [0,1] random variables. We set $X_0 = e$ and $Y_0 = 0$. If X_k and Y_k are known, we construct

$$\begin{aligned} X_{k+1} &= x_i && \text{if } \sum_{\ell=1}^{i-1} \omega(x, x_\ell) \leq u_k < \sum_{\ell=1}^{i} \omega(x, x_\ell)\,, \\ X_{k+1} &= \overleftarrow{x} && \text{otherwise}\,, \\ Y_{k+1} &= y + 2\mathbb{1}_{\{u_k \leq p\}} - 1\,, \end{aligned}$$

where $x := X_k \in \mathbb{T}$ and $y := Y_k \in \mathbb{Z}$. Then $(X_n)_{n\geq 0}$ has the distribution of our $\mathbb{T}$-RWRE indeed, and $(Y_n)_{n\geq 0}$ is a random walk on $\mathbb{Z}$ which increases of one unit with probability $p > 1/2$ and decreases of the same value with probability $1-p$. Notice also that on the event $\{D(e) = \infty\}$, we have

$$|X_{k+1}| - |X_k| \geq Y_{k+1} - Y_k\,.$$

It implies that the first regeneration time $\mathcal{R}_1$ of $(Y_n)_{n\geq 0}$ defined by

$$\mathcal{R}_1 := \inf\{k > 0\,:\, Y_\ell < Y_k\ \forall \ell < k\,, Y_m \geq Y_k\ \forall m > k\}$$

is necessarily a regeneration time for $(X_n,\, n \geq 0)$, which proves in turn that

$$\mathbb{S}^e(\Gamma_1 > n) \leq \mathbb{Q}^e(\mathcal{R}_1 > n)\,.$$

To complete the proof, we must ensure that $\mathbb{Q}^e(\mathcal{R}_1 > n)$ is exponentially small, which is done in [DPZ96] Lemma 5.1. □

2.2 The cases "$i < \nu_{min}^{-1}$, $q_1 = 0$" and " $i < \nu_{min}^{-1}$, $s < 1$"

When $i < \nu_{min}^{-1}$, if we assume also that $q_1 = 0$ or $s < 1$, we prove that Γ_1 has a subexponential tail. This situation covers, in particular, the case of RWRE on a regular tree.

Proposition 2.2. *Suppose that $i < \nu_{min}^{-1}$ and $q_1 = 0$, then there exist $1 > \alpha_1 > \alpha_2 > 0$ such that for n large enough,*

$$e^{-n^{\alpha_1}} < \mathbb{S}^e(\Gamma_1 > n) < e^{-n^{\alpha_2}}\,. \tag{2.3}$$

The same relation holds with some $1 > \alpha_3 > \alpha_4 > 0$ in the case "$i < \nu_{min}^{-1}$ and $s < 1$".

Proof of Proposition 2.2 : lower bound. We only suppose that $i < \nu_{min}^{-1}$, which allows us to deal with both cases of the lemma. Define for some $p' \in (0, 1/2)$ and $b \in \mathbb{N}$,

$$w_+ \;:=\; \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i) \geq \frac{1-p'}{p'},\, \nu(e) \leq b\right)\,,$$

$$w_- \quad := \quad \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i) \leq \frac{p'}{1-p'},\, \nu(e) \leq b\right).$$

By (1.2), $E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right] > 1$ and therefore $\mathbf{Q}\left(\sum_{i=1}^{\nu(e)} A(e_i) > 1\right) > 0$. Since ess inf $A < \nu_{min}^{-1}$, it guarantees that $\mathbf{Q}\left(\sum_{i=1}^{\nu(e)} A(e_i) < 1\right) > 0$. Consequently, by choosing p' close enough of $1/2$ and b large, we can take w_+ and w_- positive. Let $c := \frac{1}{6\ln(b)}$, and define $h_n := \lfloor c\ln(n)\rfloor$. A tree $\mathbb{T}$ is said to be n-good if

- any vertex x of the h_n first generations verifies $\nu(x) \leq b$ and $\sum_{i=1}^{\nu(x)} A(x_i) \geq \frac{1-p'}{p'}$,
- any vertex x of the h_n following generations verifies $\nu(x) \leq b$ and $\sum_{i=1}^{\nu(x)} A(x_i) \leq \frac{p'}{1-p'}$.

We observe that $\mathbf{Q}(\mathbb{T}$ is n-good$) \geq w_+^{h_n b^{h_n}} w_-^{h_n b^{2h_n}} \geq e^{-n^{1/3+o(1)}}$ which is stretched exponential, i.e. behaving like $e^{-n^{r+o(1)}}$ for some $r \in (0,1)$. Define the events

$$\begin{aligned}
E_1 &:= \{\text{at time } \tau_{h_n} \text{ we can't find an edge of level smaller than } h_n \text{ crossed only once}\} \\
&\quad\ \cap \{D(e) > \tau_{h_n}\}, \\
E_2 &:= \{\text{the walk visits the level } h_n\ n \text{ times before reaching the root or the level } 2h_n\}, \\
E_3 &:= \{\text{after the } n\text{-th visit of level } h_n, \text{ the walk reaches level } 2h_n \text{ before level } h_n\}, \\
E_4 &:= \{\text{after time } \tau_{2h_n} \text{ the walk never comes back to level } 2h_n - 1\}.
\end{aligned}$$

Suppose that the tree is n-good. Since A is supposed bounded, there exists a constant $c_1 > 0$ such that for any x neighbour of y, we have

$$\omega(x,y) \geq \frac{c_1}{\nu(x)}. \tag{2.4}$$

It yields that $P_\omega^e(E_1)^{-1} = O(n^K)$ for some $K > 0$ (where $O(n^K)$ means that the function is bounded by a factor of $n \to n^K$). Combine (2.4) with the strong Markov property at time τ_{h_n} to see that

$$P_\omega^e(E_3 \,|\, E_1 \cap E_2)^{-1} = O(n^K),$$

where K is taken large enough. We emphasize that the functions $O(n^K)$ are deterministic. Still by Markov property,

$$P_\omega^e(E_1 \cap E_2 \cap E_3 \cap E_4) = E_\omega^e[\mathbb{1}_{E_1\cap E_2\cap E_3}\beta(X_{\tau_{2h_n}})]. \tag{2.5}$$

Let $(Y'_n)_{n\geq 0}$ be the random walk on $\mathbb{Z}$ starting from zero with

$$P_\omega(Y'_{n+1} = k+1 \,|\, Y'_n = k) = 1 - P_\omega(Y'_{n+1} = k-1 \,|\, Y'_n = k) = p'.$$

We introduce $T'_i := \inf\{k \geq 0 \,:\, Y_k = i\}$, and p'_n the probability that $(Y'_n)_{n\geq 0}$ visits h_n before -1 :

$$p'_n := P_\omega(T'_{-1} < T'_{h_n})\,.$$

By a coupling argument similar to that encountered in the proof of Proposition 2.1, we show that in an n-good tree,

$$P^e_\omega(E_1 \cap E_2) \;\geq\; P^e_\omega(E_1)(p'_n)^n \;=\; O(n^K)^{-1}(p'_n)^n\,, \tag{2.6}$$

which gives

$$P^e_\omega(E_1 \cap E_2 \cap E_3) \geq O(n^K)^{-1}(p'_n)^n\,. \tag{2.7}$$

Observing that $\mathbb{Q}^e(\Gamma_1 > n,\, D(e) = \infty) \geq E_{\mathbf{Q}}\left[\mathbb{1}_{\{\mathbb{T}\text{ is }n\text{-good}\}}\mathbb{1}_{E_1\cap E_2\cap E_3\cap E_4}\right]$, we obtain by (2.5)

$$\begin{aligned}\mathbb{Q}^e(\Gamma_1 > n,\, D(e) = \infty) &\geq E_{\mathbb{Q}^e}\left[\mathbb{1}_{\{\mathbb{T}\text{ is n-good}\}}\mathbb{1}_{E_1\cap E_2\cap E_3}\beta(X_{\tau_{2h_n}})\right]\\ &= E_{\mathbb{Q}^e}\left[\mathbb{1}_{\{\mathbb{T}\text{ is n-good}\}}P^e_\omega(E_1\cap E_2\cap E_3)\right]E_{\mathbf{Q}}[\beta]\,,\end{aligned}$$

by independence. By (2.7),

$$\mathbb{Q}^e(\Gamma_1 > n,\, D(e) = \infty) \geq O(n^K)^{-1}\mathbf{Q}\,(\mathbb{T}\text{ is }n\text{-good})\,(p'_n)^n\,.$$

We already know that $\mathbf{Q}\,(\mathbb{T}$ is n-good$)$ has a stretched exponential lower bound, and it remains to observe that the same holds for $(p'_n)^n$. But the method of gambler's ruin shows that $p'_n \geq 1 - \left(\frac{p'}{1-p'}\right)^{h_n}$, which gives the required lower bound by our choice of h_n. □

Let us turn to the upper bound. We divide the proof in two, depending on which case we deal with.

Proof of Proposition 2.2 : upper bound in the case $q_1 = 0$. Assume that $q_1 = 0$ (the condition $i < \nu_{min}^{-1}$ is not required in the proof). The proof of the following lemma is deferred. Recall the notation introduced in (2.2), $\gamma(e) := P^e_\omega(T_{\overleftarrow{e}} = T^*_e = \infty) \leq \beta(e)$.

Lemma 2.3. *When* $q_1 = 0$, *there exists a constant* $c_2 \in (0,1)$ *such that for large* n,

$$E_{\mathbf{Q}}\left[(1-\gamma(e))^n\right] \leq e^{-n^{c_2}}\,.$$

Denote by π_k the k-th distinct site visited by the walk $(X_n,\, n \geq 0)$. We observe that

$$\begin{aligned}\mathbb{Q}^e(\Gamma_1 > n^3) &\leq \mathbb{Q}^e(\Gamma_1 > \tau_n) + \mathbb{Q}^e(\text{ more than } n^2 \text{ distinct sites are visited before } \tau_n) \\ &\quad + \mathbb{Q}^e(\exists\, k \leq n^2 \,:\, N(\pi_k) > n).\end{aligned} \tag{2.8}$$

Since $\mathbb{Q}^e(\Gamma_1 > \tau_n) = \mathbb{Q}^e(|X_{\Gamma_1}| > n)$, it follows from Fact A that $\mathbb{Q}^e(\Gamma_1 > \tau_n)$ decays exponentially. For the second term of the right-hand side, beware that

$$\begin{aligned}&\mathbb{Q}^e(\text{ more than } n^2 \text{ distinct sites are visited before } \tau_n) \\ \leq\ &\sum_{k=1}^{n} \mathbb{Q}^e(\text{more than } n \text{ distinct sites are visited at level } k)\,.\end{aligned}$$

If we denote by t_i^k the first time when the i-th distinct site of level k is visited, we have, by the strong Markov property,

$$\begin{aligned}P_\omega^e\,(\text{more than } n \text{ sites are visited at level } k) &= P_\omega^e\left(t_n^k < \infty\right) \\ &\leq P_\omega^e\left(t_{n-1}^k < \infty,\, D\left(X_{t_{n-1}^k}\right) < \infty\right) \\ &= E_\omega^e\left[\mathbb{1}_{\{t_{n-1}^k < \infty\}}\left(1 - \beta(X_{t_{n-1}^k})\right)\right]\,.\end{aligned}$$

The independence of the environments entails that

$$E_{\mathbb{Q}^e}\left[\mathbb{1}_{\{t_{n-1}^k < \infty\}}\left(1 - \beta(X_{t_{n-1}^k})\right)\right] = \mathbb{Q}^e\left(t_{n-1}^k < \infty\right) E_{\mathbf{Q}}[1-\beta]\,.$$

Consequently,

$$\begin{aligned}\mathbb{Q}^e\left(t_n^k < \infty\right) &\leq \mathbb{Q}^e\left(t_{n-1}^k < \infty\right) E_{\mathbf{Q}}[1-\beta] \\ &\leq \left(E_{\mathbf{Q}}[1-\beta]\right)^{n-1}\,,\end{aligned} \tag{2.9}$$

which leads to

$$\mathbb{Q}^e\left(\text{more than } n^2 \text{ sites are visited before } \tau_n\right) \leq n\left(E_{\mathbf{Q}}[1-\beta]\right)^{n-1}\,, \tag{2.10}$$

which is exponentially small. We remark, for later use, that equation (2.9) holds without the assumption $q_1 = 0$. For the last term of equation (2.8), we write

$$\mathbb{Q}^e\left(\exists\, k \leq n^2 \,:\, N(\pi_k) > n\right) \leq \sum_{k=1}^{n^2} \mathbb{Q}^e\left(N(\pi_k) > n\right)\,.$$

Let $U := \bigcup_{n\geq 0}(\mathbb{N}^*)^n$ be the set of words, where $(\mathbb{N})^0 := \{\emptyset\}$. Each vertex x of $\mathbb{T}$ is naturally associated with a word of U, and $\mathbb{T}$ is then a subset of U (see [Nev86] for a more complete description). For any $k \geq 1$,

$$\begin{aligned}\mathbb{Q}^e(N(\pi_k) > n) &= \sum_{x\in U} \mathbb{Q}^e\left(x \in \mathbb{T},\, N(x) > n,\, x = \pi_k\right) \\ &\leq \sum_{x\in U} E_{\mathbf{Q}}\left[\mathbb{1}_{\{x\in\mathbb{T}\}} P^e_\omega(x = \pi_k)(1-\gamma(x))^n\right],\end{aligned}$$

with the notation of (2.2). By independence,

$$\begin{aligned}\mathbb{Q}^e(N(\pi_k) > n) &\leq \sum_{x\in U} E_{\mathbf{Q}}\left[\mathbb{1}_{\{x\in\mathbb{T}\}} P^e_\omega(x = \pi_k)\right] E_{\mathbf{Q}}\left[(1-\gamma(e))^n\right] \\ &= E_{\mathbf{Q}}[(1-\gamma(e))^n].\end{aligned}$$

Apply Lemma 2.3 to complete the proof. □

Proof of Lemma 2.3. Let $\mu > 0$ be such that $q := \mathbf{Q}(\beta(e) > \mu) > 0$, and write

$$R := \inf\{k \geq 1 \,:\, \exists |x| = k,\, \beta(x) \geq \mu\}.$$

Let x_R be such that $|x_R| = R$ and $\beta(x_R) \geq \mu$ and we suppose for simplicity that x_R is a descendant of e_1. We see that $\gamma(e) \geq \omega(e, e_1)\beta(e_1) \geq \frac{c_1}{\nu(e)}\beta(e_1)$ by equation (2.4). In turn, equation (2.1) of [Aid08] implies that for any vertex x, we have

$$\frac{1}{\beta(x)} = 1 + \frac{1}{\sum_{i=1}^{\nu(x)} A(x_i)\beta(x_i)} \leq 1 + \frac{1}{\text{ess inf } A}\frac{1}{\beta(x_i)},$$

for any $1 \leq i \leq \nu(x)$. By recurrence on the path from e_1 to x_R, this leads to

$$\frac{1}{\beta(e_1)} \leq 1 + \frac{1}{\text{ess inf } A} + \ldots + \left(\frac{1}{\text{ess inf } A}\right)^{R-1}\frac{1}{\mu}.$$

We deduce the existence of constants c_4, $c_5 > 0$ such that

$$\gamma(e) \geq \frac{c_4}{\nu(e)} e^{-c_5 R}. \tag{2.11}$$

It yields that

$$E_{\mathbf{Q}}\left[(1-\gamma(e))^n\, \mathbb{1}_{\{\nu(e)<\sqrt{n}\}}\right] \leq \mathbf{Q}\left(R > \frac{1}{4c_5}\ln(n)\right) + e^{-n^{1/4+o(1)}}.$$

We observe that

$$\mathbf{Q}\left(R > \frac{1}{4c_5}\ln(n)\right) \leq \mathbf{Q}\left(\forall |x| = \frac{1}{4c_5}\ln(n),\, \beta(x) > \mu\right).$$

By assumption, $q_1 = 0$; thus $\#\{x \in \mathbb{T} : |x| = \frac{1}{4c_5}\ln(n)\} \geq 2^{1/4c_5 \ln(n)} =: n^{c_6}$. As a consequence, $\mathbf{Q}\left(\forall |x| = \frac{1}{4c_5}\ln(n),\, \beta(x) > \mu\right) \leq q^{n^{c_6}}$. Hence, the proof of our lemma is reduced to find a stretched exponential bound for $E_{\mathbf{Q}}\left[(1-\gamma(e))^n \mathbb{1}_{\{\nu(e)\geq\sqrt{n}\}}\right]$. For any $x \in \mathbb{T}$, denote by V_x^μ the number of children x_i of x such that $\beta(x_i) > \mu$. For $\varepsilon \in (0, \mathbf{Q}(\beta(e) > \mu))$,

$$\begin{aligned} & E_{\mathbf{Q}}\left[(1-\gamma(e))^n \mathbb{1}_{\{\nu(e)\geq\sqrt{n}\}}\right] \\ \leq\ & \mathbb{Q}^e\left(\nu(e) \geq \sqrt{n},\, V_e^\mu < \varepsilon\nu(e)\right) + E_{\mathbf{Q}}\left[(1-\gamma(e))^n \mathbb{1}_{\{V_e^\mu \geq \varepsilon\nu(e)\}}\right]. \end{aligned}$$

We apply Cramér's Theorem to handle with the first term on the right-hand side. Turning to the second one, the bound is clear once we observe the general inequality,

$$\gamma(e) = \sum_{k=1}^{\nu(e)} \omega(e, e_k)\beta(e_k) \geq \frac{c_1}{\nu(e)}\sum_{k=1}^{\nu(e)} \beta(e_k) \geq \frac{c_1\mu}{\nu(e)} V_e^\mu, \tag{2.12}$$

which is greater than $c_1\mu\varepsilon$ on $\{V_e^\mu \geq \varepsilon\nu(e)\}$. □

Remark 2.3. As a by-product, we obtain that $E_{\mathbf{Q}}\left[(1-\gamma(e))^n \mathbb{1}_{\{\nu(e)\geq\sqrt{n}\}}\right] \leq e^{-n^{c_3}}$ without the assumption $q_1 = 0$.

Proof of Proposition 2.2 : upper bound in the case $s < 1$. We follow the strategy of the case "$q_1 = 0$". The proof boils down to the estimate of

$$\begin{aligned} & \mathbb{Q}^e(N(\pi_k) > n,\, D(e) = \infty) \\ =\ & \mathbb{Q}^e(N(\pi_k) > n,\, \nu(\pi_k) < \sqrt{n},\, D(e) = \infty) + \mathbb{Q}^e(N(\pi_k) > n,\, \nu(\pi_k) \geq \sqrt{n},\, D(e) = \infty). \end{aligned}$$

Let $x \in \mathbb{T}$ and consider the RWRE $(X_n,\, n \geq 0)$ when starting from $\overleftarrow{x}$. Inspired by Lyons et al. [LPP96], we propose to couple it with a random walk $(Y_n'',\, n \geq 0)$ on $\mathbb{Z}$. We first define X_n'' as the restriction of X_n on the path $[\![\overleftarrow{e}, x]\!]$. Beware that X_n'' exists only up to a time T, which corresponds to the time when the walk $(X_n,\, n \geq 0)$ escapes the path $[\![\overleftarrow{e}, x]\!]$, id est leaves the path and never comes back to it. After this time, we set $X_n'' = \Delta$ for some Δ in some space $\mathcal{E}$. Then $(X_n'')_{n\geq 0}$ is a random walk on $[\![\overleftarrow{e}, x]\!] \cup \{\Delta\}$, whose transition probabilities are, if $y \notin \{\overleftarrow{e},\, x,\, \Delta\}$,

$$\begin{aligned}
P_\omega^{\overleftarrow{x}}(X''_{n+1} = y_+ \,|\, X''_n = y) &= \frac{\omega(y,y_+)}{\omega(y,y_+) + \omega(y,\overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y,y_k)\beta(y_k)}\,, \\
P_\omega^{\overleftarrow{x}}(X''_{n+1} = \overleftarrow{y} \,|\, X''_n = y) &= \frac{\omega(y,\overleftarrow{y})}{\omega(y,y_+) + \omega(y,\overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y,y_k)\beta(y_k)}\,, \\
P_\omega^{\overleftarrow{x}}(X''_{n+1} = \Delta \,|\, X''_n = y) &= \frac{\sum_{k=1}^{\nu(y)} \omega(y,y_k)\beta(y_k)}{\omega(y,y_+) + \omega(y,\overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y,y_k)\beta(y_k)}\,,
\end{aligned}$$

where y_+ is the child of y which lies on the path $[\![\overleftarrow{e}, x]\!]$. Besides, the walk is absorbed on Δ and reflected on $\overleftarrow{e}$ and x. We recall that $s := \operatorname{ess\,sup} A$. We construct the adequate coupling with a biased random walk $(Y''_n)_{n\geq 0}$ on $\mathbb{Z}$, starting from $|x|-1$, increasing with probability $s/(1+s)$, decreasing otherwise and such that $Y''_n \geq |X''_n|$ as long as $X''_n \neq \Delta$ (which is always possible since $P_\omega(X''_{n+1} = y_+ \,|\, X''_n = y) \leq \frac{s}{1+s}$). After time T, we let Y_n move independently. By coupling and then by gambler's ruin method, it leads to

$$P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}}) \leq P_\omega^{|x|-1}(\,\exists\, n \geq 0 \,:\, Y''_n = |x|\,) = s\,.$$

It follows that

$$1 - P_\omega^x(T_x^* < T_{\overleftarrow{e}}) \geq \omega(x,\overleftarrow{x})\left(1 - P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})\right) \geq \frac{c_1(1-s)}{\nu(x)}\,,$$

by equation (2.4). Hence,

$$\begin{aligned}
&\mathbb{Q}^e(N(\pi_k) > n,\, \nu(\pi_k) \leq \sqrt{n},\, D(e) = \infty) \\
= &\sum_{x\in U} E_{\mathbf{Q}}\left[\mathbb{1}_{\{\nu(x)\leq\sqrt{n}\}} P_\omega^e\left(x = \pi_k,\, D(e) > T_x\right) P_\omega^x\left(N(x) > n, D(e) = \infty\right)\right] \\
\leq &\sum_{x\in U} E_{\mathbf{Q}}\left[P_\omega^e(x = \pi_k)\left(1 - \frac{c_1(1-s)}{\sqrt{n}}\right)^n\right] = \left(1 - \frac{c_1(1-s)}{\sqrt{n}}\right)^n\,,
\end{aligned}$$

which decays stretched exponentially. On the other hand,

$$\begin{aligned}
&\mathbb{Q}^e(N(\pi_k) > n,\, \nu(\pi_k) \geq \sqrt{n},\, D(e) = \infty) \\
\leq\ &\mathbb{Q}^e\left(\nu(\pi_k) \geq \sqrt{n},\, V_{\pi_k}^\mu < \varepsilon\nu(\pi_k)\right) + \mathbb{Q}^e\left(N(\pi_k) > n,\, V_{\pi_k}^\mu \geq \varepsilon\nu(\pi_k)\right)\,.
\end{aligned}$$

with the notation introduced in the proof of Lemma 2.3. We have

$$\mathbb{Q}^e\left(\nu(\pi_k) \geq \sqrt{n},\, V_{\pi_k}^\mu < \varepsilon\nu(\pi_k)\right) \;=\; \mathbf{Q}\left(\nu(e) \geq \sqrt{n},\, V_e^\mu < \varepsilon\nu(e)\right)\,,$$

which is stretched exponential by Cramér's Theorem. We also observe that

$$\begin{aligned}\mathbb{Q}^e\left(N(\pi_k) > n,\, V^\mu_{\pi_k} \geq \varepsilon\nu(\pi_k)\right) &\leq E_{\mathbb{Q}^e}\left[\mathbb{1}_{\{V^\mu_{\pi_k}\geq\varepsilon\nu(x)\}}(1-\gamma(\pi_k))^n\right] \\ &= E_{\mathbf{Q}}\left[\mathbb{1}_{\{V^\mu_e\geq\varepsilon\nu(x)\}}(1-\gamma(e))^n\right] \leq (1-c\mu\varepsilon)^n\ ,\end{aligned}$$

by equation (2.12). This completes the proof. □

2.3 The case $\Lambda < \infty$

In this part, we suppose that $\Lambda < \infty$, where Λ is defined by

$$\Lambda := Leb\left\{t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{q_1}\right\}.$$

We prove that the tail distribution of Γ_1 is polynomial.

Proposition 2.4. *If $\Lambda < \infty$, then*

$$\lim_{n\to\infty}\frac{1}{\ln(n)}\ln\left(\mathbb{S}^e(\Gamma_1 > n)\right) = -\Lambda\,. \tag{2.13}$$

Proof of Proposition 2.4. Lemma 3.3 of [Aid08] already gives

$$\liminf_{n\to\infty}\frac{1}{\ln(n)}\ln\left(\mathbb{S}^e(\Gamma_1 > n)\right) \geq -\Lambda.$$

Hence, the lower bound of (2.13) is known. The rest of the section is dedicated to the proof of the upper bound.

We start with three preliminary lemmas. We first prove an estimate for one-dimensional RWRE, that will be useful later on. Denote by $(R_n,\, n\geq 0)$ a generic RWRE on $\mathbb{Z}$ such that the random variables $A(i)$, $i \geq 0$ are independent and have the distribution of A, when we set for $i \geq 0$,

$$A(i) := \frac{\omega_R(i,i+1)}{\omega_R(i,i-1)}$$

with $\omega_R(y,z)$ the quenched probability to jump from y to z. We denote by $P^k_{\omega,R}$ the quenched distribution associated with $(R_n,\, n\geq 0)$ when starting from k, and by $\mathbf{P}_R$ the distribution of the environment ω_R. Let $c_7 \in (0,1)$ be a constant whose value will be given later on. For any $k \geq \ell \geq 0$ and $n \geq 0$, we introduce the notation

$$p(\ell,k,n) := E_{\mathbf{P}_R}\left[(1-c_7P^\ell_{\omega,R}(T^*_\ell > T_0\wedge T_k))^n\right]\,. \tag{2.14}$$

Lemma 2.5. *Let* $0 < r < 1$, *and* $\Lambda_r := Leb\left\{t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{r}\right\}$. *Then, for any* $\varepsilon > 0$, *we have for* n *large enough,*

$$\sum_{k \geq \ell \geq 0} r^k p(\ell, k, n) \leq n^{-\Lambda_r + \varepsilon}.$$

Proof. The method used is very similar to that of Lemma 5.1 in [Aid08]. We feel free to present a sketch of the proof. We consider the one-dimensional RWRE $(R_n)_{n\geq 0}$. We introduce for $k \geq \ell \geq 0$, the potential $V(0) = 0$ and

$$\begin{aligned} V(\ell) &= -\sum_{i=0}^{\ell-1} \ln(A(i)), \\ H_1(\ell) &= \max_{0\leq i\leq \ell} V(i) - V(\ell), \\ H_2(\ell, k) &= \max_{\ell\leq i\leq k} V(i) - V(\ell). \end{aligned}$$

We know (e.g. [Zei04]) that

$$\frac{e^{-H_2(\ell+1,k)}}{k+1} \leq P_{\omega,R}^{\ell+1}\left(T_k < T_\ell\right) \leq e^{-H_2(\ell+1,k)}, \tag{2.15}$$

$$\frac{e^{-H_1(\ell)}}{k+1} \leq P_{\omega,R}^{\ell-1}\left(T_{-1} < T_\ell\right) \leq e^{-H_1(\ell)}. \tag{2.16}$$

It yields that

$$P_{\omega,R}^{\ell}(T_\ell^* > T_0 \wedge T_k) \geq e^{-H_1(\ell)\wedge H_2(\ell,k) + O(\ln k)},$$

where $O(\ln k)$ is a deterministic function. Let $\eta \in (0,1)$.

$$\begin{aligned} p(\ell, k, n) &\leq (1 - c_7 n^{-1+\eta})^n + \mathbf{P}_R(H_1(\ell) \wedge H_2(\ell, k) - O(\ln k) \geq (1-\eta)\ln(n)) \\ &\leq e^{-c_8 n^\eta} + \mathbf{P}_R(H_1(\ell) \wedge H_2(\ell, k) - O(\ln k) \geq (1-\eta)\ln(n)). \end{aligned}$$

In Section 8.1 of [Aid08], we proved that for any $s \in (0,1)$, $E_{\mathbf{P}_R}\left[e^{\Lambda_s (H_1(\ell)\wedge H_2(\ell,k))}\right] \leq e^{k\ln(1/s) + o_s(k)}$, where $o_s(k)$ is such that $o_s(k)/k$ tends to 0 at infinity. This implies that, defining $\widetilde{o}_s(k) := o_s(k) - \Lambda_s O(\ln k)$,

$$\begin{aligned} & s^k \mathbf{P}_R\left(H_1(\ell) \wedge H_2(\ell, k) - O(\ln k) \geq (1-\eta)\ln(n)\right) \\ \leq\ & s^k \left(1 \wedge e^{k\ln(1/s) - \Lambda_s(1-\eta)\ln(n) + \widetilde{o}_s(k)}\right) \\ \leq\ & n^{-\Lambda_s(1-\eta)} \exp\left((k\ln(s) + \Lambda_s(1-\eta)\ln(n)) \wedge \widetilde{o}_s(k)\right). \end{aligned}$$

Observe that there exists M_s such that for any k and any n, we have $(k\ln(s)+\Lambda_s(1-\eta)\ln(n))\wedge\widetilde{o}_s(k)\leq\sup_{i\leq M_s\ln(n)}\widetilde{o}(i)+\eta\ln n$, and notice that $\sup_{i\leq M_s\ln(n)}\widetilde{o}_s(i)$ is negligible towards $\ln(n)$. This leads to, for n large enough,

$$s^k p(\ell,k,n)\leq s^k e^{-c_8 n^\eta}+n^{-\Lambda_s(1-\eta)+2\eta}.$$

Let $r\in(0,1)$ and $s>r$. We have

$$r^k p(\ell,k,n)\leq r^k e^{-c_8 n^\eta}+\left(\frac{r}{s}\right)^k n^{-\Lambda_s(1-\eta)+2\eta}.$$

Lemma 2.5 follows by choosing η small enough and s close enough to r. □

Let Z_n represent the size of the n-th generation of the tree $\mathbb{T}$. We have the following result.

Lemma 2.6. *There exists a constant $c_9>0$ such that for any $H>0$, $B>0$ and n large enough,*

$$E_{\mathbf{Q}}\left[(1-\gamma(e))^n\,\mathbb{1}_{\{Z_H>B\}}\right]\leq n^{-c_9 B}.$$

Proof. We have

$$\begin{aligned}E_{\mathbf{Q}}\left[(1-\gamma(e))^n\,\mathbb{1}_{\{Z_H>B\}}\right]&\leq E_{\mathbf{Q}}\left[(1-\gamma(e))^n\mathbb{1}_{\{\nu(e)\geq\sqrt{n}\}}\right]+E_{\mathbf{Q}}\left[(1-\gamma(e))^n\,\mathbb{1}_{\{Z_H>B,\,\nu(e)\leq\sqrt{n}\}}\right]\\&\leq e^{-n^{c_3}}+E_{\mathbf{Q}}\left[(1-\gamma(e))^n\,\mathbb{1}_{\{Z_H>B,\,\nu(e)\leq\sqrt{n}\}}\right]\end{aligned}$$

by Remark 2.3. When $\nu(e)\leq\sqrt{n}$, we have, by (2.11),

$$\gamma(e)\geq\frac{c_4}{\sqrt{n}}e^{-c_5 R},$$

with $R:=\inf\{k\geq 1\,:\,\exists|x|=k,\,\beta(x)\geq\mu\}$ as before ($\mu>0$ is such that $q:=\mathbf{Q}(\beta(e)>\mu)>0$). Thus,

$$E_{\mathbf{Q}}\left[(1-\gamma(e))^n\,\mathbb{1}_{\{Z_H>B,\,\nu(e)\leq\sqrt{n}\}}\right]\leq\mathbf{Q}\left(R>\frac{1}{4c_5}\ln(n)+H,\,Z_H>B\right)+e^{-n^{1/4+o(1)}}.$$

By considering the Z_H subtrees rooted at each of the individuals in generation H, we see that

$$\begin{aligned}\mathbf{Q}\left(R>c_{10}\ln(n)+H,\,Z_H>B\right)&=E_{GW}\left[\mathbf{Q}(R>c_{10}\ln(n))^{Z_H}\,\mathbb{1}_{\{Z_H>B\}}\right]\\&\leq\mathbf{Q}(R>c_{10}\ln(n))^B.\end{aligned}$$

If $R > c_{10}\ln(n)$, we have in particular $\beta(x) < \mu$ for each $|x| = c_{10}\ln(n)$ which implies that

$$\mathbf{Q}\left(R > c_{10}\ln(n) + H,\, Z_H > B\right) \leq E_{GW}\left[q^{Z_{c_{10}\ln(n)}}\right]^B .$$

Let $t \in (q_1, 1)$. For n large enough, $E_{GW}\left[q^{Z_{c_{10}\ln(n)}}\right] \leq t^{c_{10}\ln(n)} = n^{c_{10}\ln(t)}$, $(E_{GW}[q^{Z_n}]/q_1^n$ has a positive limit by Corollary 1 page 40 of [AN72]). The lemma follows. $\square$

Let $r \in (q_1, 1)$, $\varepsilon > 0$, B be such that

$$c_9 B\varepsilon > 2\Lambda \tag{2.17}$$

and H large enough so that

$$GW(Z_H \leq B) < r^H \frac{1}{B} < 1. \tag{2.18}$$

In particular, $c_{11} := GW(Z_H > B) > 0$.

Let $\nu(x,k)$ denote for any $x \in \mathbb{T}$ the number of descendants of x at generation $|x| + k$ ($\nu(x,1) = \nu(x)$), and let

$$\mathcal{S}_H := \{x \in \mathbb{T} \,:\, \nu(x,H) > B\}. \tag{2.19}$$

For any $x \in \mathbb{T}$, we call $F(x)$ the youngest ancestor of x which lies in $\mathcal{S}_H$, and $G(x)$ an oldest descendant of x in $\mathcal{S}_H$. For any $x, y \in \mathbb{T}$, we write $x \leq y$ if y is a descendant of x and $x < y$ if besides $x \neq y$. We define for any $x \in \mathbb{T}$, $W(x)$ as the set of descendants y of x such that there exists no vertex z with $x < z \leq y$ and $\nu(z,H) > B$. In other words, $W(x) = \{y \,:\, y \geq x,\, F(y) \leq x\}$. We define also

$$\begin{aligned}
\overset{\circ}{W}(x) &:= W(x)\backslash\{x\}, \\
\partial W(x) &:= \{y \,:\, \overleftarrow{y} \in W(x),\, \nu(y,H) > B\}.
\end{aligned}$$

Finally, let $W_j(e) := \{x \,:\, |x| = j,\, x \in W(e)\}$.

Lemma 2.7. *Recall that $m := E_{GW}[\nu(e)]$ and r is a real belonging to $(q_1, 1)$. We also recall that H and B verify $GW(Z_H \leq B) < r^H \frac{1}{B}$. We have for any $j \geq 0$,*

$$E_{GW}\left[W_j(e)\right] < m\, r^{j-1} .$$

Proof. We construct the subtree $\mathbb{T}_H$ of the tree $\mathbb{T}$ by retaining only the generations kH, $k \geq 0$ of the tree $\mathbb{T}$. Let

$$\mathbb{W} = \mathbb{W}(\mathbb{T}) := \{x \in \mathbb{T}_H : \forall y \in \mathbb{T}_H, (y < x) \Rightarrow \nu(y, H) \leq B\}. \tag{2.20}$$

The tree $\mathbb{W}$ is a Galton–Watson tree whose offspring distribution is of mean $E_{GW}[Z_H \mathbb{1}_{\{Z_H \leq B\}}] \leq B \times GW(Z_H \leq B) \leq r^H$ by (2.18). Then for each child e_i of e (in the original tree $\mathbb{T}$), let $\mathbb{W}^i := \mathbb{W}(\mathbb{T}_{e_i})$ where $\mathbb{T}_{e_i}$ is the subtree rooted at e_i. We conclude by observing that $W_j(e) \leq \sum_{i=1}^{\nu(e)} \#\{x \in \mathbb{W}^i \,:\, |x| = 1 + \lceil (j-1)/H \rceil \times H\}$ hence $E_{GW}[W_j(e)] \leq E_{GW}[\nu(e)] r^{j-1}$. □

We still have $r \in (q_1, 1)$ and $\varepsilon > 0$. We prove that for n large enough, and r and ε close enough to q_1 and 0, we have

$$\mathbb{Q}^e\left(\Gamma_1 > n,\, D(e) = \infty\right) \leq c_{12} n^{-(1-2\varepsilon)\Lambda_r + 3\varepsilon}, \tag{2.21}$$

where $\Lambda_r := Leb\{t \in \mathbb{R} \,:\, \mathbf{E}[A^t] \leq \frac{1}{r}\}$ as in Lemma 2.5. This suffices to prove Proposition 2.4 since ε and Λ_r can be arbitrarily close to 0 and Λ, respectively. We recall that we defined B, H and $\mathcal{S}_H$ in (2.17),(2.18) and (2.19).

The strategy is to divide the tree in subtrees in which vertices are constrained to have a small number of children (at most B children at generation H). With $B = H = 1$, we would have literally pipes. In general, the traps constructed are slightly larger than pipes. We then evaluate the time spent in such traps by comparison with a one-dimensional random walk.

We define π_k^s as the k-th distinct site visited in the set $\mathcal{S}_H$. We observe that

$$\begin{aligned}
(2.22)\quad & \mathbb{Q}^e\left(\Gamma_1 > n,\, D(e) = \infty\right) \\
\leq\ & \mathbb{Q}^e\left(\Gamma_1 > \tau_{\ln^2(n)}\right) + \mathbb{Q}^e\left(\text{more than } \ln^4(n) \text{ distinct sites are visited before } \tau_{\ln^2(n)}\right) \\
&+\ \mathbb{Q}^e\left(\exists\, k \leq \ln^4(n),\, \exists\, x \in W(\pi_k^s),\, N(x) > n/\ln^4(n)\right) \\
&+\ \mathbb{Q}^e\left(\exists\, x \in W(e),\, N(x) > n/\ln^4(n),\, D(e) = \infty,\, Z_H \leq B\right).
\end{aligned}$$

The first term on the right-hand side decays like $e^{-\ln^2(n)}$ by Fact A, and so does the second term by equation (2.9). We proceed to estimate the third term on the right-hand side of (2.22). Since

$$\mathbb{Q}^e\left(\exists k \leq \ln^4(n), \exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)\right) \leq \sum_{k=1}^{\ln^4(n)} \mathbb{Q}^e\left(\exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)\right)$$

we look at the rate of decay of $\mathbb{Q}^e\left(\exists\, x \in W(\pi_k^s), N(x) > n/\ln^4(n)\right)$ for any $k \geq 1$. We first show that the time spent at the frontier of $W(\pi_k^s)$ will be negligible. Precisely, we show

$$\mathbb{Q}^e\left(N(\pi_k^s) > n^\varepsilon\right) \leq c_{14} n^{-2\Lambda}\,, \tag{2.23}$$

$$\mathbb{Q}^e\left(\exists\, z \in \partial W(\pi_k^s),\, N(z) > n^\varepsilon\right) \leq c_{15} n^{-2\Lambda}\,. \tag{2.24}$$

As $P_\omega^y(N(y) > n^\varepsilon) \leq (1-\gamma(y))^{n^\varepsilon}$ for any $y \in \mathbb{T}$, we have,

$$\begin{aligned} \mathbb{Q}^e\left(N(\pi_k^s) > n^\varepsilon\right) &= E_{\mathbf{Q}}\left[\sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) P_\omega^y(N(y) > n^\varepsilon)\right] \\ &\leq E_{\mathbf{Q}}\left[\sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y)(1-\gamma(y))^{n^\varepsilon}\right]. \end{aligned} \tag{2.25}$$

We would like to split the expectation $E_{\mathbf{Q}}\left[P_\omega^e(\pi_k^s = y)(1-\gamma(y))^{n^\varepsilon}\right]$ in two. However the random variable $P_\omega^e(\pi_k^s = y)$ depends on the structure of the first H generations of the subtree rooted at y. Nevertheless, we are going to show that, for some $c_{14} > 0$,

$$E_{\mathbf{Q}}\left[P_\omega^e(\pi_k^s = y)(1-\gamma(y))^{n^\varepsilon}\right] \leq c_{14} E_{\mathbf{Q}}\left[P_\omega^e(\pi_k^s = y)\right] E_{\mathbf{Q}}\left[(1-\gamma(y))^{n^\varepsilon} \,|\nu(y,H) > B\right]\,.$$

Let $U := \bigcup_{n\geq 0} (\mathbb{N}^*)^n$ be, as before, the set of words. We have seen that U allows us to label the vertices of any tree (see [Nev86]). Let $y \in U$ and let ω_y represent the restriction of the environment ω to the outside of the subtree rooted at y (when y belongs to the tree). For $1 \leq L \leq H$, we denote by y_L the ancestor of y such that $|y_L| = |y| - L$. We attach to each y_L the variable $\zeta(y_L) := \mathbb{1}_{\{\nu(y_L,H)>B\}}$. We notice that there exists a measurable function f such that $P_\omega^e(\pi_k^s = y) = f(\omega_y, \zeta)\mathbb{1}_{\{\nu(y,H)>B\}}$ where $\zeta := (\zeta(y_L))_{1\leq L\leq H}$. Let $\mathcal{E}(\omega_y) := \{e \in \{0,1\}^H \,:\, \mathbf{Q}(\zeta = e \,|\, \omega_y) > 0\}$. We have

$$E_{\mathbf{Q}}\left[f(\omega_y,\zeta)\,|\,\omega_y\right] \geq \max_{e \in \mathcal{E}(\omega_y)} f(\omega_y, e)\mathbf{Q}\left(\zeta \,=\, e \,|\, \omega_y\right)\,.$$

We claim that there exists a constant $c_{13} > 0$ such that for almost every ω and any $e \in \mathcal{E}(\omega_y)$,

$$\mathbf{Q}\left(\zeta = e \,|\, \omega_y\right) \geq c_{13}\,.$$

Let us prove the claim. If ω_y is such that $\nu(\overleftarrow{y}) > B$, then $\mathcal{E}(\omega_y) = \{(1,\ldots,1)\}$ and $\mathbf{Q}\left(\zeta = e \,|\, \omega_y\right) = 1$. Therefore suppose $\nu(\overleftarrow{y}) \leq B$ and let $h := \max\{1 \leq L \leq H \,:\, \nu(y_L, L) \leq B\}$. We observe that, for any $e \in \mathcal{E}(\omega_y)$, we necessarily have $e_L = 1$ for $h < L \leq H$. We are reduced to the study of

$$\mathbf{Q}\left(\zeta = e \,|\, \omega_y\right) = \mathbf{Q}\left(\bigcap_{1\leq L\leq h} \{\zeta(y_L) = e_L\} \,\Big|\, \omega_y\right)\,.$$

For any tree $\mathcal{T}$, we denote by $\mathcal{T}^j$ the restriction to the j first generations. Let also $\mathbb{T}_{y_h}$ designate the subtree rooted at y_h in $\mathbb{T}$. Since $\nu(y_h, h) \le B$, we observe that $\mathbb{T}_{y_h}^h$ belongs almost surely to a finite (deterministic) set in the space of all trees. We construct the set

$$\begin{aligned}\Psi(\mathbb{T}_{y_h}^h, e) := \{\text{tree } \mathcal{T} \,:\, \mathcal{T}^h = \mathbb{T}_{y_h}^h,\, GW(\mathcal{T}^{h+H}) > 0,\, \forall |x| \le 2H,\, \nu_{\mathcal{T}}(x) \le B \\ \forall\, 1 \le L \le h,\, \nu_{\mathcal{T}}(y_L, h) > B \text{ if and only if } e_L = 1\}\,.\end{aligned}$$

We observe that $\Psi(\mathbb{T}_{y_K}^K, e) \neq \emptyset$ as soon as $e \in \mathcal{E}(\omega_y)$. Let $\widetilde{\Psi}(\mathbb{T}_{y_K}^K, e) := \{\mathcal{T}^{h+H},\, \mathcal{T} \in \Psi(\mathbb{T}_{y_h}^h, e)\}$ be the same set but where the trees are restricted to the first $h+H$ generations. Since $\widetilde{\Psi}(\mathbb{T}_{y_K}^K, e)$ is again included in a finite deterministic set in the space of trees, we deduce that there exists $c_{13} > 0$ such that, almost surely,

$$\inf\{GW(\mathcal{T}^{h+H} \,|\, \mathcal{T}^h),\, \mathcal{T} \in \Psi(\mathbb{T}_{y_h}^h, e),\, e \in \mathcal{E}(\omega_y)\} \ge c_{13}\,.$$

Consequently,

$$\mathbf{Q}\left(\zeta = e \,|\, \omega_y\right) \ge \mathbf{Q}(\mathbb{T}_{y_h}^{h+H} \in \widetilde{\Psi}(\mathbb{T}_{y_h}^h, e) \,|\, \omega_y) \ge c_{13}\,,$$

as required. We get

$$E_{\mathbf{Q}}\left[f(\omega_y, \zeta) \,|\, \omega_y\right] \;\ge\; c_{13} \max_{e \in \mathcal{E}(\omega_y)} f(\omega_y, e) \;\ge\; c_{13}\, f(\omega_y, \zeta)\,.$$

Finally we obtain, with $c_{14} := \frac{1}{c_{13}}$,

$$f(\omega_y, \zeta) \le c_{14}\, E_{\mathbf{Q}}\left[f(\omega_y, \zeta) \,|\, \omega_y\right]\,.$$

By (2.25), it entails that

$$\begin{aligned}\mathbb{Q}^e\left(N(\pi_k^s) > n^\varepsilon\right) &\le c_{14} \sum_{y \in U} E_{\mathbf{Q}}\left[\mathbb{1}_{\{\nu(y,H)>B\}} E_{\mathbf{Q}}\left[f(\omega_y, \zeta) \,|\, \omega_y\right] (1-\gamma(y))^{n^\varepsilon}\right] \\ &= c_{14} \sum_{y \in U} E_{\mathbf{Q}}\left[f(\omega_y, \zeta)\right] E_{\mathbf{Q}}\left[\mathbb{1}_{\{\nu(e,H)>B\}} (1-\gamma(e))^{n^\varepsilon}\right] \\ &= c_{14} \sum_{y \in U} E_{\mathbf{Q}}\left[P_\omega^e(\pi_k^s = y)\right] E_{\mathbf{Q}}\left[(1-\gamma(e))^{n^\varepsilon} \,|\, \nu(e,H) > B\right]\,.\end{aligned}$$

It implies that

$$\mathbb{Q}^e\left(N(\pi_k^s) > n^\varepsilon\right) \le c_{14} E_{\mathbf{Q}}\left[(1-\gamma(e))^{n^\varepsilon} \,|\, Z_H > B\right] \le c_{14} n^{-c_9 \varepsilon B}\,,$$

by Lemma 2.6. Since $c_9\varepsilon B > 2\Lambda$, this leads to, for n large,

$$\mathbb{Q}^e\left(N(\pi_k^s) > n^\varepsilon\right) \le c_{14} n^{-2\Lambda}$$

which is equation (2.23). Similarly, recalling that $\partial W(y)$ designates the set of vertices z such that $\overleftarrow{z} \in W(y)$ and $\nu(z,H) > B$, we have that

$$\begin{aligned}
&\mathbb{Q}^e\left(\exists\, y \in \partial W(\pi_k^s),\, N(y) > n^\varepsilon\right)\\
\le\ & E_{\mathbf{Q}}\left[\sum_{y\in\mathcal{S}_H} P_\omega^e(\pi_k^s = y) \sum_{z\in\partial W(y)} (1-\gamma(z))^{n^\varepsilon}\right]\\
\le\ & c_{14} E_{\mathbf{Q}}\left[\sum_{y\in\mathcal{S}_H} P_\omega^e(\pi_k^s = y)\right] E_{GW}\left[\partial W(e)\right] E_{\mathbf{Q}}\left[(1-\gamma(e))^{n^\varepsilon} \mid Z_H > B\right]\\
=\ & c_{14} E_{GW}\left[\partial W(e)\right] E_{\mathbf{Q}}\left[(1-\gamma(e))^{n^\varepsilon} \mid Z_H > B\right].
\end{aligned}$$

We notice that $E_{GW}[\partial W] \le E_{GW}\left[\sum_{x\in W(e)} \nu(x)\right] = m E_{GW}\left[W(e)\right]$ which is finite by Lemma 2.7. It yields, by Lemma 2.6,

$$\mathbb{Q}^e\left(\exists\, x \in W(\pi_k^s),\, N(G(x)) > n^\varepsilon\right) \le c_{15} n^{-2\Lambda}$$

thus proving (2.24). Our next step is then to find an upper bound to the probability to spend most of our time at a vertex x belonging to some $\overset{\circ}{W}(y)$. To this end, recall that $G(x)$ is an oldest descendant of x such that $\nu(x,H) > B$. We have just proved that the time spent at $y(=F(x))$ or $G(x)$ is negligible. Therefore, starting from x, the probability to spend much time in x is not far from the probability to spend the same time without reaching y neither $G(x)$. Then, this probability is bound by coupling with a one-dimensional random walk.

Define $\widetilde{T}_x^{(\ell)}$ as the ℓ-th time the walk visits x after visiting either $F(x)$ or $G(x)$, id est $\widetilde{T}_x^{(1)} = T_x$ and,

$$\widetilde{T}_x^{(\ell)} := \inf\{k > \widetilde{T}_x^{(\ell-1)} \,:\, X_k = x,\, \exists\, i \in (\widetilde{T}_x^{(\ell-1)}, k),\, X_i = F(x) \text{ or } G(x)\}\,.$$

Let also $N^{(\ell)}(x) = \sum_{k=\widetilde{T}^{(\ell)}(x)}^{\widetilde{T}^{(\ell+1)}(x)-1} \mathbb{1}_{\{X_k = x\}}$ be the time spent at x between $\widetilde{T}^{(\ell)}$ and $\widetilde{T}^{(\ell+1)}$. We observe that, for any $k \ge 1$,

$$\begin{aligned}
&\mathbb{Q}^e\left(\exists\, x \in W(\pi_k^s), N(x) > n/\ln^4(n)\right)\\
\le\ & \mathbb{Q}^e\left(N(\pi_k^s) > n^\varepsilon\right) + \mathbb{Q}^e\left(\exists\, x \in W(\pi_k^s),\, N(G(x)) > n^\varepsilon\right)
\end{aligned}$$

$$+\, \mathbb{Q}^e\left(\exists\, x \in \overset{\circ}{W}(\pi_k^s),\, \exists\, \ell \le 2n^\varepsilon,\, N^{(\ell)}(x) > n^{1-2\varepsilon}\right)$$

$$(2.26) \qquad \le \quad (c_{14}+c_{15})n^{-2\Lambda} + \sum_{\ell \le 2n^\varepsilon} \mathbb{Q}^e\left(\exists\, x \in \overset{\circ}{W}(\pi_k^s),\, N^{(\ell)}(x) > n^{1-2\varepsilon}\right).$$

Since

$$\mathbb{Q}^e(\exists\, x \in W(\pi_k^s),\, N^{(\ell)}(x) > n^{1-2\varepsilon}) \le E_{\mathbf{Q}}\left[\sum_{y\in\mathcal{S}_H} P_\omega^e(\pi_k^s = y) \sum_{x\in \overset{\circ}{W}(y)} P_\omega^x(N^{(\ell)}(x) > n^{1-2\varepsilon})\right],$$

and by the strong Markov property at $\widetilde{T}_x^{(\ell)}$,

$$\begin{aligned}
P_\omega^x\left(N^{(\ell)}(x) > n^{1-2\varepsilon}\right) &= P_\omega^x\left(\widetilde{T}_x^{(\ell)} < \infty\right) P_\omega^x\left(N^{(1)}(x) > n^{1-2\varepsilon}\right) \\
&\le P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon}),
\end{aligned}$$

this yields

$$\begin{aligned}
&\mathbb{Q}^e(\exists\, x \in W(\pi_k^s),\, N^{(\ell)}(x) > n^{1-2\varepsilon}) \\
\le\;& E_{\mathbf{Q}}\left[\sum_{y\in\mathcal{S}_H} P_\omega^e(\pi_k^s = y) \sum_{x\in \overset{\circ}{W}(y)} P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon})\right] \\
\le\;& c_{14}\, E_{\mathbf{Q}}\left[\sum_{y\in\mathcal{S}_H} P_\omega^e(\pi_k^s = y)\right] E_{\mathbf{Q}}\left[\sum_{x\in \overset{\circ}{W}(e)} P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon}) \,\Big|\, Z_H > B\right] \\
(2.27) \qquad =\;& c_{14}\, E_{\mathbf{Q}}\left[\sum_{x\in \overset{\circ}{W}(e)} P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon}) \,\Big|\, Z_H > B\right].
\end{aligned}$$

For any $x \in W(e)$, define, for any $y \in [\![e, G(x)]\!]$,

$$\begin{aligned}
\widetilde{\omega}(y, y_+) &:= \frac{\omega(y,y_+)}{\omega(y,y_+) + \omega(y, \overleftarrow{y})}, \\
\widetilde{\omega}(y, \overleftarrow{y}) &:= \frac{\omega(y,\overleftarrow{y})}{\omega(y,y_+) + \omega(y, \overleftarrow{y})},
\end{aligned}$$

where as before y_+ represents the child of y on the path. We let $(\widetilde{X}_n)_{n\ge 0}$ be the random walk on $[\![e, G(x)]\!]$ with the transition probabilities $\widetilde{\omega}$ and we denote by $\widetilde{P}_{\omega,x}(\cdot)$ the probability distribution of $(\widetilde{X}_n,\, n \geq 0)$. By Lemma 4.4 of [Aid08], we have the following comparisons :

$$\begin{aligned} P_\omega^{\overleftarrow{x}}(T_x < T_e) &\leq \widetilde{P}_{\omega,x}^{\overleftarrow{x}}(T_x < T_e)\,, \\ P_\omega^{x_+}(T_{G(x)} < T_x) &\leq \widetilde{P}_{\omega,x}^{x_+}(T_{G(x)} < T_x)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} & P_\omega^x(T_x^* < T_e \wedge T_{G(x)}) \\ = \;& \omega(x,\overleftarrow{x})P_\omega^{\overleftarrow{x}}(T_x < T_e) + \omega(x,x_+)P_\omega^{x_+}(T_x < T_{G(x)}) + \sum_{i \leq \nu(x):x_i \neq x^+} \omega(x,x_i)(1-\beta(x_i)) \\ \leq \;& \omega(x,\overleftarrow{x})\widetilde{P}_{\omega,x}^{\overleftarrow{x}}(T_x < T_e) + \omega(x,x_+)\widetilde{P}_{\omega,x}^{x_+}(T_x < T_{G(x)}) + \sum_{i \leq \nu(x):x_i \neq x_+} \omega(x,x_i) \\ = \;& 1 - \Big(\omega(x,\overleftarrow{x}) + \omega(x,x_+)\Big)\,\widetilde{P}_{\omega,x}^x(T_x^* > T_e \wedge T_{G(x)})\,. \end{aligned}$$

Since $\nu(x) \leq B$ (for $x \in \overset{\circ}{W}(e)$), we find by (2.4) a constant $c_{16} \in (0,1)$ such that $\omega(x,\overleftarrow{x}) + \omega(x,x_+) \geq c_{16}$. It yields that

$$P_\omega^x(T_x^* < T_e \wedge T_{G(x)}) \leq 1 - c_{16}\widetilde{P}_{\omega,x}^x(T_x^* > T_e \wedge T_{G(x)})\,.$$

We observe that, for any $x \in W(e)$, with the notation of (2.14) and taking $c_7 := c_{16}$,

$$E_{\mathbf{P}}\left[\Big(1 - c_{16}\widetilde{P}_{\omega,x}^x(T_x^* > T_e \wedge T_{G(x)})\Big)^n\right] = p(|x|, |G(x)|, n)\,.$$

It follows that

$$E_{GW}\left[\sum_{x \in \overset{\circ}{W}(e)} \mathbb{P}^x(N^{(1)}(x) > n^{1-2\varepsilon})\right] \leq E_{GW}\left[\sum_{x \in \overset{\circ}{W}(e)} p(|x|, |G(x)|, n^{1-2\varepsilon})\right]\,.$$

On the other hand, $\sum_{x \in W(e)} p(|x|, |G(x)|, n^{1-2\varepsilon}) \leq \sum_{y \in \partial W(e)} \sum_{x \leq y} p(|x|, |y|, n^{1-2\varepsilon})$. It implies that

$$\begin{aligned} E_{GW}\left[\sum_{x \in \overset{\circ}{W}(e)} \mathbb{P}^x(N^{(1)}(x) > n^{1-2\varepsilon})\right] &\leq \sum_{j \geq 0} E_{GW}\left[\#\{y \in \partial W(e), |y| = j\}\right]\left(\sum_{i \leq j} p(i,j,n^{1-2\varepsilon})\right) \\ &\leq m \sum_{j \geq 0} E_{GW}\left[W_{j-1}(e)\right]\left(\sum_{i \leq j} p(i,j,n^{1-2\varepsilon})\right)\,. \end{aligned}$$

By Lemmas 2.5 and 2.7, for n large enough,

$$E_{GW}\left[\sum_{x\in \mathring{W}(e)} \mathbb{P}^x(N^{(1)}(x)>n^{1-2\varepsilon})\right] \leq m^2\sum_{j\geq 0} r^{j-2}\left(\sum_{i\leq j} p(i,j,n^{1-2\varepsilon})\right) \leq n^{-(1-2\varepsilon)\Lambda_r+\varepsilon}. \tag{2.28}$$

Supposing r and ε close enough to q_1 and 0, equation (2.28) combined with (2.26) and (2.27), shows that, for any $k\geq 1$,

$$\mathbb{Q}^e\left(\exists\, x\in W(\pi_k^s), N(x)>n/\ln^4(n)\right) \leq c_{17} n^{-(1-2\varepsilon)\Lambda_r+2\varepsilon}.$$

We arrive at

$$\mathbb{Q}^e\left(\exists\, k\leq \ln^4(n),\, \exists\, x\in W(\pi_k^s),\, N(x)>n/\ln^4(n)\right) \leq c_{18} n^{-(1-2\varepsilon)\Lambda_r+3\varepsilon}. \tag{2.29}$$

Finally, the estimate of $\mathbb{Q}^e\left(\exists\, x\in W(e),\, N(x)>n/\ln^4(n),\, D(e)=\infty,\, Z_H\leq B\right)$ in (2.22) is similar. Indeed,

$$\begin{aligned}
&\mathbb{Q}^e\left(\exists\, x\in W(e),\, N(x)>n/\ln^4(n),\, D(e)=\infty,\, Z_H\leq B\right)\\
\leq\ &\mathbb{Q}^e\left(N(e)>n^\varepsilon,\, D(e)=\infty,\, \nu(e)\leq B\right) + \mathbb{Q}^e\left(\exists\, x\in W(e),\, N(G(x))>n^\varepsilon\right)\\
&+\mathbb{Q}^e\left(\exists\, x\in W(e),\, \exists\, \ell\leq 2n^\varepsilon,\, N^{(\ell)}(x)>n^{1-2\varepsilon}\right).
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{Q}^e\left(N(e)>n^\varepsilon,\, D(e)=\infty,\, \nu(e)\leq B\right) &\leq E_{\mathbf{Q}}\left[(1-\omega(e,\overleftarrow{e}))^{n^\varepsilon}\mathbb{I}_{\{\nu(e)\leq B\}}\right]\\
&\leq (1-c_1/B)^{n^\varepsilon},
\end{aligned}$$

by (2.4). By equation (2.24),

$$\mathbb{Q}^e\left(\exists\, x\in W(\pi_k^s),\, N(G(x))>n^\varepsilon\right)\leq c_{15} n^{-2\Lambda}.$$

Finally,

$$\begin{aligned}
\mathbb{Q}^e\left(\exists\, x\in \mathring{W}(e),\, \exists\, \ell\leq 2n^\varepsilon,\, N^{(\ell)}(x)>n^{1-2\varepsilon}\right) &\leq \sum_{\ell\leq 2n^\varepsilon}\mathbb{Q}^e\left(\exists\, x\in \mathring{W}(e),\, N^{(\ell)}(x)>n^{1-2\varepsilon}\right)\\
&\leq 2n^\varepsilon\mathbb{Q}^e\left(\exists\, x\in \mathring{W}(e),\, N^{(1)}(x)>n^{1-2\varepsilon}\right)\\
&\leq 2n^\varepsilon E_{GW}\left[\sum_{x\in\mathring{W}(e)}\mathbb{P}^x(N^{(1)}(x)>n^{1-2\varepsilon})\right]\\
&\leq c_{17} n^{-(1-2\varepsilon)\Lambda_r+2\varepsilon},
\end{aligned}$$

by (2.28). We deduce that, for n large enough,

(2.30) $\mathbb{Q}^e\left(\exists\, x \in W(e),\, N(x) > n/\ln^4(n),\, D(e)=\infty,\, Z_H \le B\right) \le n^{-(1-2\varepsilon)\Lambda_r+3\varepsilon}$.

In view of (2.22) combined with (2.29) and (2.30), equation (2.21) is proved, and Proposition 2.4 follows. □

3 Large deviations principles

We recall the definition of the first regeneration time

$$\Gamma_1 := \inf\left\{k > 0\, :\nu(X_k) \ge 2,\, D(X_k)=\infty,\, k=\tau_{|X_k|}\right\}.$$

We define by iteration

$$\Gamma_n := \inf\left\{k > \Gamma_{n-1}\, :\nu(X_k) \ge 2,\, D(X_k)=\infty,\, k=\tau_{|X_k|}\right\}$$

for any $n \ge 2$. We have the following fact (points (i) to (iii) are already discussed in [Aid08]; point (iv) is shown in [Gro04] in the case of regular trees and in [LPP96] in the case of biased random walks, and is easily adaptable to our case).

Fact B

(i) *For any* $n \ge 1$, $\Gamma_n < \infty \quad \mathbb{Q}^e$*-a.s.*

(ii) *Under* $\mathbb{Q}^e$, $(\Gamma_{n+1}-\Gamma_n, |X_{\Gamma_{n+1}}| - |X_{\Gamma_n}|)$, $n \ge 1$ *are independent and distributed as* $(\Gamma_1, |X_{\Gamma_1}|)$ *under the distribution* $\mathbb{S}^e$.

(iii) *We have* $E_{\mathbb{S}^e}[\,|X_{\Gamma_1}|\,] < \infty$.

(iv) *The speed* v *verifies* $v = \frac{E_{\mathbb{S}^e}[\,|X_{\Gamma_1}|\,]}{E_{\mathbb{S}^e}[\,\Gamma_1\,]}$.

The rest of the section is devoted to the proof of Theorems 1.1 and 1.2. It is in fact easier to prove them when conditioning on never returning to the root. Our theorems become

Theorem 3.1. (Speed-up case) *There exist two continuous, convex and strictly decreasing functions* $I_a \le I_q$ *from* $[1, 1/v]$ *to* $\mathbb{R}_+$, *such that* $I_a(1/v) = I_q(1/v) = 0$ *and for* $a < b$, $b \in [1, 1/v]$,

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n} \in]a,b] \,\Big|\, D(e)=\infty\right)\right) = -I_a(b)\,, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(\frac{\tau_n}{n} \in]a,b] \,\Big|\, D(e)=\infty\right)\right) = -I_q(b)\,. \tag{3.2}$$

Theorem 3.2. (Slowdown case) *There exist two continuous, convex functions $I_a \leq I_q$ from $[1/v, +\infty[$ to $\mathbb{R}_+$, such that $I_a(1/v) = I_q(1/v) = 0$ and for any $1/v \leq a < b$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\frac{\tau_n}{n} \in [a, b[\,\Big|\, D(e) = \infty\right)\right) \;=\; -I_a(a)\,, \tag{3.3}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(\frac{\tau_n}{n} \in [a, b[\,\Big|\, D(e) = \infty\right)\right) \;=\; -I_q(a)\,. \tag{3.4}$$

If ess inf $A =: i > \nu_{min}^{-1}$, then I_a and I_q are strictly increasing on $[1/v, +\infty[$. If $i \leq \nu_{min}^{-1}$, then $I_a = I_q = 0$.

Theorems 1.1 and 1.2 follow from Theorems 3.1 and 3.2 and the following proposition.

Proposition 3.3. *We have, for $a < b \leq 1/v$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e(\frac{\tau_n}{n} \in]a, b])\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e(\frac{\tau_n}{n} \in]a, b] \,|\, D(e) = \infty)\right)\,, \tag{3.5}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega(\frac{\tau_n}{n} \in]a, b])\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega(\frac{\tau_n}{n} \in]a, b] \,|\, D(e) = \infty)\right)\,. \tag{3.6}$$

Similarly, in the slowdown case, we have for $1/v \leq a < b$,

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e(\frac{\tau_n}{n} \in [a, b[)\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e(\frac{\tau_n}{n} \in [a, b[\,|\, D(e) = \infty)\right)\,, \tag{3.7}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega(\frac{\tau_n}{n} \in [a, b[)\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega(\frac{\tau_n}{n} \in [a, b[\,|\, D(e) = \infty)\right)\,. \tag{3.8}$$

Theorems 3.1 and 3.2 are proved in two distinct parts for sake of clarity. Proposition 3.3 is proved in subsection 3.3.

3.1 Proof of Theorem 3.1

For any real numbers $h \geq 0$ and $b \geq 1$, any integer $n \in \mathbb{N}$ and any vertex $x \in \mathbb{T}$ with $|x| = n$, define

$$\begin{aligned} A(h, b, x) &:= \{\omega \,:\, P^e_\omega\left(\tau_n = T_x,\, \tau_n \leq bn,\, T_{\overleftarrow{e}} > \tau_n\right) \geq e^{-hn}\}\,, \\ e_n(h, b) &:= E_{\mathbf{Q}}\left[\sum_{|x|=n} \mathbb{1}_{A(h,b,x)}\right]\,. \end{aligned}$$

We define also for any $b \geq 1$

$$h_c(b) \;:=\; \inf\{h \geq 0 \,:\, \exists\, p \in \mathbb{N},\, e_p(h, b) > 0\}\,.$$

Lemma 3.4. *There exists for any $b \geq 1$ and $h > h_c(b)$, a real $e(h,b) > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln(e_n(h,b)) = \ln(e(h,b)) .$$

Moreover, the function $(h,b) \to \ln(e(h,b))$ from $\{(h,b) \in \mathbb{R}_+ \times [1,+\infty[:\ h > h_c(b)\}$ to $\mathbb{R}$ is concave, is nondecreasing in h and in b, and

$$\lim_{h\to\infty} \ln(e(h,b)) = \ln(m) .$$

Proof. Let $x \leq y$ be two vertices of $\mathbb{T}$ with $|x| = n$ and $|y| = n+m$. We observe that

$$\begin{aligned} A(h,b,y) &\supset A(h,b,x) \cap \{\omega \,:\, P^x_\omega(\tau_{n+m} = T_y,\, \tau_{n+m} \leq bm,\, T_{\overleftarrow{x}} > \tau_{n+m}) \geq e^{-hm}\} \\ &=: A(h,b,x) \cap A_x(h,b,y). \end{aligned}$$

It yields that

$$\begin{aligned} e_{n+m}(h,b) &\geq E_{\mathbf{Q}}\left[\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} \sum_{|y|=n+m, y\geq x} \mathbb{1}_{A_x(h,b,y)}\right] \\ &= E_{\mathbf{Q}}\left[\sum_{|x|=n} \mathbb{1}_{A(h,b,x)}\right] E_{\mathbf{Q}}\left[\sum_{|x|=m} \mathbb{1}_{A(h,b,x)}\right] \\ &= e_n(h,b) e_m(h,b) . \end{aligned} \tag{3.9}$$

Let $h > h_c$ and p be such that $e_p(h_c, b) > 0$, where we write h_c for $h_c(b)$. Then $e_{np}(h_c, b) > 0$ for any $n \geq 1$. We want to show that $e_k(h,b) > 0$ for k large enough. By (2.4), $\omega(e, e_1) \geq c_1$ if $\nu(e) = 1$ so that $e_k(-\ln(c_1), b) \geq q_1^k$. Let n_c be such that $e^{-h_c n_c} c_1 \geq e^{-h n_c}$. We check as before that for any $n \geq n_c$, and any $r \leq p$, we have indeed

$$\begin{aligned} e_{np+r}(h,b) &\geq e_{np}(h_c,b) e_r(-\ln(c_1), b) \\ &\geq e_{np}(h_c,b) q_1^r > 0 . \end{aligned}$$

Thus (3.9) implies that

$$\lim_{n\to\infty} \frac{1}{n} \ln(e_n(h,b)) = \sup\left\{\frac{1}{k} \ln(e_k(h,b)),\ k \geq 1\right\} =: \ln(e(h,b)) , \tag{3.10}$$

with $e(h,b) > 0$. Similarly, we can check that

$$e_n(th_1 + (1-t)h_2, tb_1 + (1-t)b_2) \geq e_{nt}(h_1, b_1) e_{n(1-t)}(h_2, b_2) ,$$

which leads to

$$\ln(e(th_1 + (1-t)h_2, tb_1 + (1-t)b_2)) \geq t\ln(e(h_1, b_1)) + (1-t)\ln(e(h_2, b_2)),$$

hence the concavity of $(h,b) \to \ln(e(h,b))$. The fact that $e(h,b)$ is nondecreasing in h and in b is direct. Finally, $\limsup_{h\to\infty} \ln(e(h,b)) \leq \ln(m)$ and $\liminf_{h\to\infty} \ln(e(h,b)) \geq \liminf_{h\to\infty} \ln(e_1(h,b)) = \ln(m)$ by dominated convergence. □

In the rest of the section, we extend $e(h,b)$ to $\mathbb{R}_+ \times [1,+\infty[$ by taking $e(h,b) = 0$ for $h \leq h_c(b)$.

Corollary 3.5. *Let $S := \{h \geq 0 : e(h,b) > 1\}$ and $S' := \{h \geq 0 : e(h,b) \geq 1\}$. We have*

$$\sup\{e^{-h}\, e(h,b),\, h \in S\} = \sup\{e^{-h}\, e(h,b),\, h \in S'\}.$$

Proof. Let $M := \inf\{h : e(h,b) > 1\}$. We claim that if $h < M$, then $e(h,b) < 1$. Indeed, suppose that there exists $h_0 < M$ such that $e(h_0,b) \geq 1$. Then $e(h_0,b) = 1$ by definition of M, so that $e(h,b)$ is constant equal to 1 on $[h_0, M[$. By concavity, $\ln(e(h,b))$ is equal to 0 on $[h_0,+\infty[$, which is impossible since it tends to $\ln(m)$ at infinity. The corollary follows. □

We have the tools to prove Theorem 1.1.

Proof of Theorem 1.1. For $b \in [1,+\infty[$, let

$$\begin{aligned} J_a(b) &:= -\sup\{-h + \ln(e(h,b)),\, h \geq 0\}, \\ J_q(b) &:= -\sup\{-h + \ln(e(h,b)),\, h \in S\}. \end{aligned}$$

Define then for any $b \leq 1/v$,

$$\begin{aligned} I_a(b) &= J_a(b), \\ I_q(b) &= J_q(b). \end{aligned}$$

We immediately see that $I_a \leq I_q$. The convexity of J_a and J_q stems from the convexity of the function $h - \ln(e(h,b))$. Indeed, let J represent either J_a or J_q and let $1 \leq b_1 \leq b_2$ and $t \in [0,1]$. Denote by h_1, h_2, b and h the reals that verify

$$J(b_1) = h_1 - \ln(e(h_1, b_1)),$$

$$\begin{aligned} J(b_2) &= h_2 - \ln(e(h_2, b_2))\,, \\ h &:= th_1 + (1-t)h_2\,, \\ b &:= tb_1 + (1-t)b_2\,. \end{aligned}$$

We observe that

$$\begin{aligned} J(b) &\leq h - \ln(e(h,b)) \\ &\leq t(h_1 - \ln(e(h_1,b_1))) + (1-t)(h_2 - \ln(e(h_2,b_2))) = tJ(b_1) + (1-t)J(b_2) \end{aligned}$$

which proves the convexity. We show now that, for any $b \geq 1$,

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \leq bn\right)\right) = -J_a(b)\,, \tag{3.11}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \leq bn\right)\right) = -J_q(b)\,. \tag{3.12}$$

We first prove (3.11). Since $\mathbb{Q}^e\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \leq bn\right) \geq e^{-hn} e_n(h,b)$ for any $h \geq 0$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e(\tau_n < T_{\overleftarrow{e}},\, \tau_n \leq bn)\right) \geq -I_a(b).$$

Turning to the upper bound, take a positive integer k. We observe that

$$\begin{aligned} \mathbb{Q}^e\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \leq bn\right) &\leq \sum_{\ell=0}^{k-1} e^{-n\ell/k} e_n\left((\ell+1)/k, b\right) \\ &\leq k e^{n/k} \sup\{e^{-hn} e_n(h,b),\, h \geq 0\}\,. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \leq bn\right)\right) \leq \frac{1}{k} - J_a(b)\,.$$

Letting k tend to infinity gives the upper bound of (3.11).

To prove equation (3.12), let k be still a positive integer and $h \in S$. Denote by $V_{pk}(\mathbb{T})$ the set of vertices $|x| = pk$ such that $P_\omega^{x_{\ell-1}}\left(\tau_{\ell k} < T_{\overleftarrow{x}_{\ell-1}},\, \tau_{\ell k} = T_{x_\ell} \leq bk\right) \geq e^{-hk}$ for any $\ell \leq p$, where x_ℓ represents the ancestor of x at generation ℓk. Call $V(\mathbb{T}) := \cup_{p\geq 0} V_{pk}(\mathbb{T})$ the subtree thus obtained. We observe that V is a Galton–Watson tree of mean offspring $e_k(h,b)$. Let

$$\mathcal{T}_{k,h} := \{\mathbb{T} \,:\, V(\mathbb{T})\,\text{is infinite}\}\,.$$

Take $\mathbb{T} \in \mathcal{T}_{k,h}$. For any $x \in V_{pk}$, we have

$$\begin{aligned} & P^e_\omega\left(\tau_{pk} < T_{\overleftarrow{e}},\, \tau_{pk} = T_x \le bpk\right) \\ \ge\ & P^e_\omega\left(\tau_k < T_{\overleftarrow{e}},\, \tau_k = T_{x_1} \le bk\right) \dots P^{x_{k-1}}_\omega\left(\tau_{pk} < T_{\overleftarrow{x}_{k-1}},\, \tau_{pk} = T_x \le bk\right) \ge e^{-hpk}\,. \end{aligned}$$

It implies that

$$P^e_\omega\left(\tau_{pk} < T_{\overleftarrow{e}},\, \tau_{pk} \le bpk\right) \ge e^{-hpk} \# V_{pk}(\mathbb{T})\,.$$

By the Seneta–Heyde Theorem (see [AN72] page 30 Theorem 3),

$$\lim_{p\to\infty} \frac{1}{p} \ln\left(\# V_{pk}(\mathbb{T})\right) = \ln(e_k(h,b)) \qquad \mathbf{Q}-\text{a.s.}$$

It follows that, as long as $\mathbb{T} \in \mathcal{T}_{k,h}$,

$$\liminf_{p\to\infty} \frac{1}{pk} \ln\left(P^e_\omega\left(\tau_{pk} < T_{\overleftarrow{e}},\, \tau_{pk} \le bpk\right)\right) \ge -h + \frac{1}{k}\ln(e_k(h,b))\,.$$

Notice that

$$P^e_\omega\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \le bn\right) \ge P^e_\omega\left(\tau_{pk} < T_{\overleftarrow{e}},\, \tau_{pk} \le bpk\right) \min_{|x|=pk} P^x_\omega\left(\tau_n < T_{\overleftarrow{x}},\, \tau_n \le b(n-pk)\right)$$

where $p := \lfloor \frac{n}{k} \rfloor$. Since A is bounded, there exists $c_{17} > 0$ such that $\sum_{i=1}^{\nu(y)} \omega(y, y_i) \ge c_{17}$ $\forall y \in \mathbb{T}$. It yields that

$$\min_{|x|=pk} P^x_\omega\left(\tau_n < T_{\overleftarrow{x}},\, \tau_n = (n-pk)\right) \ge c_{17}^k\,.$$

Hence,

$$\liminf_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \le bn\right)\right) \ge -h + \frac{1}{k}\ln(e_k(h,b))\,. \tag{3.13}$$

Take now a general tree $\mathbb{T}$. Notice that since $h \in S$, $\mathbf{Q}\left(\mathcal{T}_{k,h}\right) > 0$ for k large enough, and there exists almost surely a vertex $z \in \mathbb{T}$ such that the subtree rooted at it belongs to $\mathcal{T}_{k,h}$. It implies that for large k, (3.13) holds almost surely. Then letting k tend to infinity and taking the supremum over all $h \in S$ leads to

$$\liminf_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(\tau_n < T_{\overleftarrow{e}},\, \tau_n \le bn\right)\right) \ge -J_q(b)\,.$$

For the upper bound in (3.12), we observe that, for any integer k,

$$P^e_\omega(\tau_n < T_{\overleftarrow{e}},\, \tau_n \le bn) \le \sum_{\ell=0}^{k-1} e^{-\ell n/k} \sum_{|x|=n} \mathbb{1}_{A((\ell+1)/k,b,x)}\,.$$

By Markov's inequality, we have

$$\mathbf{Q}\left(\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} > (e(h,b)+1/k)^n\right) \le \frac{e_n(h,b)}{(e(h,b)+1/k)^n} \le \left(\frac{e(h,b)}{e(h,b)+1/k}\right)^n,$$

by (3.10). An application of the Borel–Cantelli lemma proves that $\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} \le (e(h,b)+1/k)^n$ for all but a finite number of n, $\mathbf{Q}$-a.s. In particular, if $e(h,b)+1/k < 1$, then $\sum_{|x|=n} \mathbb{1}_{A(h,b,x)} = 0$ for n large enough. Consequently, for n large,

$$P_\omega^e(\tau_n < T_{\overleftarrow{e}},\, \tau_n \le bn) \le e^{n/k} k \sup\{e^{-hn}(e(h,b)+1/k)^n,\, h : e(h,b)+1/k \ge 1\}.$$

We find that

$$\limsup_{n\to\infty} \frac{1}{n} \ln(P_\omega^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn)) \le 1/k + \sup\{-h + \ln(e(h,b)+1/k),\, h : e(h,b)+1/k \ge 1\}.$$

Let k tend to infinity and use Corollary 3.5 to complete the proof of (3.12).

We observe that

$$\begin{aligned} P_\omega^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) - P_\omega^e(\tau_n < T_{\overleftarrow{e}} < \infty, \tau_n \le bn) &\le P_\omega^e(T_{\overleftarrow{e}} = \infty,\, \tau_n \le bn) \\ &\le P_\omega^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn). \end{aligned}$$

But $P_\omega^e(\tau_n < T_{\overleftarrow{e}} < \infty, \tau_n \le bn) \le P_\omega^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) \max_{i=1,\dots,\nu(e)}(1-\beta(e_i))$. Since $\max_{i=1,\dots,\nu(e)}(1-\beta(e_i)) < 1$ almost surely, we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \ln(P_\omega^e(\tau_n \le bn) \,|\, D(e) = \infty) = -J_q(b). \tag{3.14}$$

In the annealed case, notice that $\mathbb{S}^e(\tau_n < T_{\overleftarrow{e}} < \infty, \tau_n \le bn) = \mathbb{S}^e(\tau_n < T_{\overleftarrow{e}}, \tau_n \le bn) E_{\mathbf{P}}[1-\beta]$ which leads similarly to

$$\lim_{n\to\infty} \frac{1}{n} \ln(\mathbb{S}^e(\tau_n \le bn)) = -J_a(b). \tag{3.15}$$

We can now finish the proof of the theorem. The continuity has to be proved only at $b=1$ (since J_a and J_q are convex on $[1,+\infty[$), which is directly done with the arguments of [DGPZ02] Section 4. We let $b < 1/v = E_{\mathbb{S}^e}[\Gamma_1]/E_{\mathbb{S}^e}[|X_{\Gamma_1}|]$ and we observe that for any constant $c_{18} > 0$,

$$\mathbb{S}^e(\tau_n \le bn) \le \mathbb{S}^e(\tau_n < \Gamma_{c_{18}n}) + \mathbb{S}^e(\Gamma_{c_{18}n} \le bn).$$

Choose c_{18} such that $b\left(E_{\mathbb{S}^e}[\Gamma_1]\right)^{-1} < c_{18} < \left(E_{\mathbb{S}^e}[|X_{\Gamma_1}|]\right)^{-1}$. Use Cramér's Theorem with Facts A and B to see that $\mathbb{S}^e(\tau_n < \Gamma_{c_{18}n})$ and $\mathbb{S}^e(\Gamma_{c_{18}n} \le bn)$ decrease exponentially. Then, $\mathbb{S}^e(\tau_n \le bn)$ has an exponential decay and, by (3.15), $I_a(b) > 0$ which leads to $I_q(b) > 0$ since $I_a \le I_q$. We deduce in particular that I_a and I_q are strictly decreasing. Furthermore, $P^e_\omega(\tau_n \le bn \,|\, D(e) = \infty)$ tends to 1 almost surely when $b > 1/v$, which in virtue of (3.14), implies that $J_q(b) = 0$. By continuity, $I_q(1/v) = 0$ and therefore $I_a(1/v) = 0$. Finally, let $a < b$, $b \in [1, 1/v]$.

$$P^e_\omega\left(an < \tau_n \le bn \,|\, D(e) = \infty\right) = P^e_\omega\left(\tau_n \le bn \,|\, D(e) = \infty\right) - P^e_\omega\left(\tau_n \le an \,|\, D(e) = \infty\right).$$

Equation (3.2) follows since I_q is strictly decreasing. The same argument proves (3.1). □

3.2 Proof of Theorem 3.2

The proof is the same as before by taking for $b \ge 1$,

$$\begin{aligned}
\widetilde{A}(h,b,x) &:= \left\{\omega \,:\, P^e_\omega\left(\tau_n = T_x,\, T_{\overleftarrow{e}} > \tau_n \ge bn\right) \ge e^{-hn}\right\},\\
\widetilde{e}_n(h,b) &:= E_{\mathbf{Q}}\left[\sum_{|x|=n} \mathbb{1}_{\widetilde{A}(h,b,x)}\right],\\
\widetilde{S} &:= \{h \,:\, \widetilde{e}(h,b) > 1\}.
\end{aligned}$$

Define also for any $b \ge 1$,

$$\begin{aligned}
\widetilde{J}_a(b) &:= -\sup\{-h + \ln(\widetilde{e}(h,b)),\, h \ge 0\},\\
\widetilde{J}_q(b) &:= -\sup\{-h + \ln(\widetilde{e}(h,b)),\, h \in \widetilde{S}\},
\end{aligned}$$

and for any $b \ge 1/v$,

$$\begin{aligned}
I_a(b) &:= \widetilde{J}_a(b),\\
I_q(b) &:= \widetilde{J}_q(b).
\end{aligned}$$

We verify that $I_a \le I_q$ and both functions are convex. We have then for any $b \ge 1$,

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{Q}^e\left(T_{\overleftarrow{e}} > \tau_n \ge bn\right)\right) = -\widetilde{J}_a(b), \tag{3.16}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P^e_\omega\left(T_{\overleftarrow{e}} > \tau_n \ge bn\right)\right) = -\widetilde{J}_q(b). \tag{3.17}$$

As before, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{S}^e\left(\tau_n\geq bn\right)\right) &= -\widetilde{J}_a(b)\,,\\ \lim_{n\to\infty}\frac{1}{n}\ln\left(P^e_\omega\left(\tau_n\geq bn\,|\,D(e)=\infty\right)\right) &= -\widetilde{J}_q(b)\,.\end{aligned}$$

We have $\widetilde{J}_a = \widetilde{J}_q = 0$ on $[1, 1/v]$. In the case $i > \nu_{min}^{-1}$, the positivity of I_a and I_q on $]1/v, +\infty[$ comes from Proposition 2.1 and Cramér's Theorem, which implies that they are strictly increasing. Equations (3.3) and (3.4) follow in that case. In the case $i \leq \nu_{min}^{-1}$, we follow the strategy of [DGPZ02]. Let $\eta > 0$. As in the proof of Proposition 2.2, we set $h_n := \lfloor \ln(n)/(6\ln(b))\rfloor$, and for some $b \in \mathbb{N}$,

$$\begin{aligned} w_+ &:= \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i)\geq 1+\eta,\, \nu(e)\leq b\right)\,,\\ w_- &:= \mathbf{Q}\left(\sum_{i=1}^{\nu} A(e_i)\leq \frac{1}{1+\eta},\, \nu(e)\leq b\right)\,.\end{aligned}$$

Taking b large enough, we have $w_+ > 0$ and $w_- > 0$. We say that $\mathbb{T}$ is a n-good tree if

- any vertex x of the h_n first generations verifies $\nu(x)\leq b$ and $\sum_{i=1}^{\nu(x)} A(x_i)\geq 1+\eta$,
- any vertex x of the h_n following generations verifies $\nu(x)\leq b$ and $\sum_{i=1}^{\nu(x)} A(x_i)\leq \frac{1}{1+\eta}$.

Then we know that $Q_n := \mathbf{Q}(\mathbb{T}\,\text{is}\,n\text{-good}) \geq \exp(-n^{1/3+o(1)})$. Let Y' be a random walk starting from zero which increases (resp. decreases) of 1 with probability $\frac{1+\eta}{2+\eta}$ (resp. $\frac{1}{2+\eta}$). We define p'_n as the probability that Y' reaches -1 before h_n. We show that (2.6) is still true (by the exactly same arguments), so that there exists a constant $K > 0$ and a deterministic function $O(n^K)$ bounded by a factor of $n \to n^K$, such that

$$P^e_\omega(T_{\overleftarrow{e}} > \tau_{2h_n}\geq n)\geq O(n^K)^{-1}(p'_n)^n\,, \tag{3.18}$$

We have, by gambler's ruin formula,

$$p'_n = 1-\frac{1}{1+\left(\frac{1}{1+\eta}\right)+\ldots+\left(\frac{1}{1+\eta}\right)^{h_n}}\geq\frac{1}{1+\eta}\,.$$

Let $k_n := \lfloor n^d\rfloor$ with $d\in(1/3,1/2)$ and let $f\in(d,1-d)$. We call an n-slow tree a tree in which we can find a vertex $|x| = k_n$ such that $\mathbb{T}_x$ is n-good (where $\mathbb{T}_x$ is the subtree rooted at x), and for any $y\leq x$, we have $\nu(y)\leq\exp(n^f)$. We observe that if a tree is not n-slow, then either there exists a vertex before generation k_n with more than $\exp(n^f)$ children, or any subtree rooted at generation k_n is not n-good. This leads to

$$\begin{aligned}\mathbf{Q}(\mathbb{T}\,\text{is not}\,n\text{-slow}) &\leq \sum_{\ell=1}^{k_n} E_{GW}[Z_\ell]GW(\nu > e^{n^f}) + E_{GW}\left[(1-Q_n)^{Z_{k_n}}\right] \\ &\leq k_n m^{k_n} m e^{-n^f} + (1-Q_n)^{(1+\varepsilon)^{k_n}} + GW(Z_{k_n} \leq (1+\varepsilon)^{k_n})\,.\end{aligned}$$

We notice that $(1-Q_n)^{(1+\varepsilon)^{k_n}} \leq \exp(-(1+\varepsilon)^{n^{d+o(1)}})$. Moreover,

$$GW(Z_{k_n} \leq (1+\varepsilon)^{k_n}) \leq (1+\varepsilon)^{k_n} E_{GW}\left[\frac{1}{Z_{k_n}}\right]$$

Observe that for any $k \geq 0$, $E_{GW}\left[\frac{1}{Z_{k+1}}\right] \leq q_1 E_{GW}\left[\frac{1}{Z_k}\right] + (1-q_1)E_{GW}\left[\frac{1}{X_1+X_2}\right]$ where X_1 and X_2 are independent and distributed as Z_k. We then verify $E_{GW}\left[\frac{1}{X_1+X_2}\right] \leq (u/2) \wedge v$ where $u := E_{GW}\left[\min(X_1,X_2)^{-1}\right]$ and $v := E_{GW}\left[\max(X_1,X_2)^{-1}\right]$. Since $u+v = E_{GW}\left[\frac{2}{Z_k}\right]$, we deduce that $E_{GW}\left[\frac{1}{X_1+X_2}\right] \leq \frac{2}{3}E_{GW}\left[\frac{1}{Z_k}\right]$, leading to $E_{GW}\left[\frac{1}{Z_{k+1}}\right] \leq (q_1+\frac{2}{3}(1-q_1))E_{GW}\left[\frac{1}{Z_k}\right] \leq (q_1+\frac{2}{3}(1-q_1))^{k+1}$. We get

$$GW(Z_{k_n} \leq (1+\varepsilon)^{k_n}) \leq \left((1+\varepsilon)(q_1+\frac{2}{3}(1-q_1))\right)^{k_n},$$

and, taking ε small enough,

$$\mathbf{Q}(\mathbb{T}\,\text{is not}\,n\text{-slow}) \leq \exp(-n^{d+o(1)})\,. \tag{3.19}$$

Let $1/v \leq a < b$. We want to show that (under the hypothesis $i \leq \nu_{min}^{-1}$),

$$\liminf_{n\to\infty} \ln P_\omega^e(\frac{\tau_n}{n} \in [a,b[,\ D(e) > \tau_n) = 0\,. \tag{3.20}$$

If this is proved, the Jensen's inequality gives

$$\liminf_{n\to\infty} \ln \mathbb{Q}^e(\frac{\tau_n}{n} \in [a,b[,\ D(e) > \tau_n) = 0\,. \tag{3.21}$$

Equations (3.4) and (3.3) follow. Therefore, we focus on the proof of (3.20).

Let $n_1 := n - k_n - 2h_n$, $\delta > 0$, and $G_k := \{|x| = k \text{ s.t. } \mathbb{T}_x \text{ is } n\text{-slow}\}$. We have

$$\left\{\frac{\tau_n}{n} \in [a,b[,\ \tau_{\overleftarrow{e}} > \tau_n\right\} \subset E_5 \cap E_6 \cap E_7\,,$$

with

$$\begin{aligned}
E_5 &:= \left\{T_{\overleftarrow{e}} > \tau_{n_1},\, \frac{\tau_{n_1}}{n_1} \in \left[\frac{1}{v} - \delta, \frac{1}{v} + \delta\right[\right\}, \\
E_6 &:= \left\{X_{\tau_{n_1}} \in G_{n_1}\right\}, \\
E_7 &:= \left\{D(X_{\tau_{n_1}}) > \tau_n,\, \frac{\tau_n}{n} \in \left(a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta\right)\right\}.
\end{aligned}$$

We look at the probability of the event E_7 conditioned on E_5 and E_6. Therefore, we suppose that $u := X_{\tau_{n_1}}$ is known, and that the subtree $\mathbb{T}_u$ rooted at u is a n-slow tree. There exists x_n at generation $n_1 + k_n$ such that $\mathbb{T}_{x_n}$ is a n-good tree and $\nu(y) \le e^{n^f}$ for any $u \le y < x_n$. Let also n be large enough so that $k_n \le \delta n$. It implies that

$$\begin{aligned}
& P_\omega^u\left(D(u) > \tau_n,\, \frac{\tau_n}{n} \in (a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta)\right) \\
\ge\ & P_\omega^u\left(D(u) > T_{x_n} = k_n\right) P_\omega^{x_n}\left(D(x_n) > \tau_n,\, \frac{\tau_n}{n} \in (a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta)\right) \\
\ge\ & \exp(-c_{21} n^{c_{22}}) P_\omega^{x_n}\left(D(x_n) > \tau_n,\, \frac{\tau_n}{n} \in (a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta)\right),
\end{aligned}$$

for some $c_{22} \in (0,1)$. By definition of a n-good tree, any vertex x descendant of x_n and such that $|x| \le n$ verifies $\nu(x) \le b$. Therefore there exists a constant $c_{23} > 0$ such that $P_\omega^y(\tau_n \le 2h_n) \ge c_{23}^{2h_n}$ for any $y \ge x_n$, $|y| < n$. By the strong Markov property,

$$\begin{aligned}
& P_\omega^{x_n}\left(D(x_n) > \tau_n,\, \frac{\tau_n}{n} \in (a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta)\right) \\
\ge\ & P_\omega^{x_n}\left(D(x_n) > \tau_n,\, \frac{\tau_n}{n} \ge a - \frac{1}{v} + \delta\right) c_{23}^{2h_n}.
\end{aligned}$$

Let $L := a - \frac{1}{v} + \delta$. By equation (3.18),

$$P_\omega^{x_n}\left(D(x_n) > \tau_n,\, \frac{\tau_n}{n} \ge a - \frac{1}{v} + \delta\right) \ge O(n^K)^{-1}\left(\frac{1}{1+\eta}\right)^{Ln}.$$

Hence, by the strong Markov property,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \ln P_\omega^e(E_7 \,|\, E_5 \cap E_6) &= \liminf_{n\to\infty} \frac{1}{n} \ln P_\omega^u\left(D(u) > \tau_n,\, \frac{\tau_n}{n} \in (a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta)\right) \\
&\ge -L(1+\eta).
\end{aligned}$$

This implies that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \ln P_\omega^e\left(\frac{\tau_n}{n} \in [a, b[,\ D(e) > \tau_n\right) &\ge \liminf_{n\to} \frac{1}{n} \ln P_\omega^e\left(E_5 \cap E_6 \cap E_7\right) \\
&\ge \liminf_{n\to\infty} \frac{1}{n} \ln P_\omega^e\left(E_5 \cap E_6\right) - L\ln(1+\eta). \qquad (3.22)
\end{aligned}$$

Notice that

$$\begin{aligned} E_{\mathbf{Q}}\left[P_\omega^e\left(E_5 \cap E_6^c\right)\right] &= E_{\mathbf{Q}}\left[P_\omega^e\left(E_5\right) - P_\omega^e\left(E_5 \cap E_6\right)\right] \\ &= \mathbb{Q}(E_5)(1 - \mathbf{Q}(\mathbb{T}\,\text{is}\, n\text{-slow})) \\ &\leq \mathbb{Q}(E_5)\exp(-n^{d+o(1)})\,, \end{aligned}$$

by equation (3.19). By Markov's inequality,

$$\mathbf{Q}(P_\omega^e(E_5 \cap E_6^c) \geq \frac{1}{n^2}) \leq n^2 \mathbb{Q}(E_5) e^{-n^{d+o(1)}}\,.$$

The Borel–Cantelli lemma implies that almost surely, for n large enough,

$$P_\omega^e\left(E_5 \cap E_6\right) \geq P_\omega^e(E_5) - \frac{1}{n^2}\,.$$

We observe that $P_\omega^e(E_5) \to P_\omega^e(T_{\overleftarrow{e}} = \infty)$ when n goes to infinity. Therefore , equation (3.22) becomes

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\frac{\tau_n}{n} \in [a,b[,\ D(e) > \tau_n\right)\right) \geq -(a - \frac{1}{v} + \delta)\ln(1+\eta)\,.$$

We let η go to 0 to get

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(P_\omega^e\left(\frac{\tau_n}{n} \in [a,b[,\ D(e) > \tau_n\right)\right) = 0$$

which proves (3.20).

3.3 Proof of Proposition 3.3

The speed-up case is quite immediate. Indeed, reasoning on the last visit to the root, we have

$$\mathbb{Q}^e(\tau_n \leq bn,\ D(e) = \infty) \leq \mathbb{Q}^e(\tau_n \leq bn) \leq bn\mathbb{Q}^e(\tau_n \leq bn,\ D(e) = \infty)\,.$$

Therefore, by Theorem 3.1,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e(\tau_n \leq bn) = \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e(\tau_n \leq bn \,|\, D(e) = \infty)\,.$$

It already gives (3.5) since I_a is strictly decreasing on $[1, 1/v]$. We do exactly the same for the quenched inequality. Therefore, let us turn to the slowdown case, beginning with the annealed inequality (3.7). We follow the arguments of [DGPZ02]. We still write $i = \text{ess inf } A$. For technical reasons, we need to distinguish the cases where $\mathbf{P}(A = i)$ is null or positive.

We feel free to deal only with the case $\mathbf{P}(A=i)=0$, the other one following with nearly any change. Moreover, we suppose without loss of generality that $i>\nu_{min}^{-1}$, since the two sides are equal to zero when $i\le\nu_{min}^{-1}$. Let $k\ge 1$. We write $\ell=k[2]$ to say that ℓ and k have the same parity. Following [DGPZ02], we write for $b>a>1/v$,

$$\begin{aligned} & P_\omega^e(bn>\tau_n\ge an) \\ = & \sum_{\ell=k[2]}\sum_{|x|=k} P_\omega^e(bn>\tau_n\ge an,\tau_n>\ell,\, X_\ell=x, |X_i|>k,\, \forall\, i=\ell+1\ldots,\tau_n) \\ = & \sum_{\ell=k[2]}\sum_{|x|=k} P_\omega^e(\tau_n>\ell,\, X_\ell=x)P_\omega^x(bn-\ell>\tau_n>an-\ell,\, D(x)>\tau_n)\,. \end{aligned}$$

By coupling, we have, for $p:=\nu_{min}i>1$,

$$\sup_{|x|=k} P_\omega^e(\tau_n>\ell,\, X_\ell=x)\le P_\omega^e(|X_\ell|\le k)\le P(S_\ell^p\le k)\,,$$

where S_ℓ^p stands for a reflected biased random walk on the half line, which moves of $+1$ with probability $p/1+p$ and of -1 with probability $1/1+p$. From (and with the notation of) Lemma 5.2 of [DGPZ02], we know that for all ℓ of the same parity as k,

$$P(S_\ell^p\le k)\le c_k(1+\delta_k)^\ell P(S_\ell^p=k,\, 1\le S_i\le k-1,\, i=1,\ldots,\ell-1)$$

where $c_k<\infty$ and $\delta=(\delta_k)$ is a sequence independent of all the parameters and tending to zero. In particular, we stress that δ do not depend on p. Hence, $P_\omega^e(bn>\tau_n\ge an)$ is smaller than

$$c_k(1+\delta_k)^{bn}\sum_{\ell=k[2]}\sum_{|x|=k} P(S_\ell^p=k,\, 1\le S_i\le k-1,\, i=1,\ldots,\ell-1)W_n(x,\ell)$$

where

$$W_n(x,\ell):=P_\omega^x(bn-\ell>\tau_n\ge an-\ell,\, D(x)>\tau_n)\,.$$

We deduce that

$$\begin{aligned} P_\omega^e(bn>\tau_n\ge an) &\le c_k(1+\delta_k)^{bn}\sum_{\ell=k[2]}\sum_{|x|=k} P_{\omega_p}^e(\tau_k=\ell, D(e)>\ell)W_n(x,\ell) \\ &= \nu_{min}^k c_k(1+\delta_k)^{bn}\sum_{\ell=k[2]}\sum_{|x|=k} P_{\omega_p}^e(\tau_k=\ell, D(e)>\ell, X_\ell=x)W_n(x,\ell)\,, \end{aligned} \tag{3.23}$$

where ω_p represents the environment of the biased random walk on the ν_{min}-ary tree such that for any vertex x, $P_{\omega_p}^x(X_1=x_i)=\frac{p}{\nu_{min}(1+p)}$ for each child x_i, and $P_\omega^x(X_1=\overleftarrow{x})=\frac{1}{1+p}$.

Taking the expectations yields that

$$\mathbb{Q}^e(bn > \tau_n \geq an) \leq \nu_{min}^k c_k (1+\delta_k)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P^e_{\omega_p}(\tau_k = \ell, D(e) > \ell, X_\ell = x) E_{\mathbf{Q}}[W_n(x,\ell)] \,. \tag{3.24}$$

Moreover, define for any $|x| = k$,

$$\mathcal{S}^+_{k,\ell}(\mathbb{T}, x) = \left\{ \{s_i\}_{i=0}^{\ell} : |s_{i+1}| - |s_i| = 1, s_0 = 0, k-1 \geq |s_i| > 0, s_\ell = x \right\}$$

the set of paths on $\mathbb{T}$ which ends at x in ℓ steps and stays between generation 1 and $k-1$ before. We notice that, for any environment ω,

$$P^e_\omega(\tau_k = \ell, D(e) > \ell, X_\ell = x) = \sum_{\{s\} \in \mathcal{S}^+_{k,\ell}(\mathbb{T},x)} \sum_{y \in \mathbb{T}} \omega(y, \overleftarrow{y})^{N(y,\overleftarrow{y})} \sum_{i=1}^{\nu(y)} \omega(y, y_i)^{N(y,y_i)} \tag{3.25}$$

where for each path $\{s_i\}$, $N(z,y)$ stands for the number of passage from z to y. Let $\varepsilon > 0$, and $\mathcal{G}_k$ denote for any k the set of trees such that any vertex x of generation less than k verifies $\nu(x) = \nu_{min}$ and $A(x) \leq \text{ess inf } A + \varepsilon$. Let $p' := \nu_{min}(\text{ess inf A} + \varepsilon)$. We observe that

$$P^e_{\omega_p}(\tau_k = \ell, D(e) > \ell, X_\ell = x) \;\; = \;\; \sum_{\{s\} \in \mathcal{S}^+_{k,\ell}(\mathbb{T},x)} \sum_{y \in \mathbb{T}} \left(\frac{1}{1+p}\right)^{N(y,\overleftarrow{y})} \sum_{i=1}^{\nu(y)} \left(\frac{p}{\nu_{min}(1+p)}\right)^{N(y,y_i)}$$

Therefore, if $\mathbb{T}$ belongs to $\mathcal{G}_k$, we have by equation (3.25),

$$P^e_{\omega_p}(\tau_k = \ell, D(e) > \ell, X_\ell = x) \leq \left(\frac{1+p'}{1+p}\right)^{\ell} P^e_\omega(\tau_k = \ell, D(e) > \ell, X_\ell = k) \,.$$

It entails that

$$\begin{aligned}
& \mathbb{1}_{\{\mathbb{T} \in \mathcal{G}_k\}} \sum_{\ell=k[2]} \sum_{|x|=k} P^e_{\omega_p}(\tau_k = \ell, D(e) > \ell, X_\ell = x) W_n(x,\ell) \\
\leq \;\; & \mathbb{1}_{\{\mathbb{T} \in \mathcal{G}_k\}} \left(\frac{1+p'}{1+p}\right)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P^e_\omega(\tau_k = \ell, D(e) > \ell, \, X_\ell = x) W_n(x,\ell) \\
= \;\; & \mathbb{1}_{\{\mathbb{T} \in \mathcal{G}_k\}} \left(\frac{1+p'}{1+p}\right)^{bn} P^e_\omega(bn > \tau_n \geq an, \, D(e) > \tau_n) \\
\leq \;\; & \left(\frac{1+p'}{1+p}\right)^{bn} P^e_\omega(bn > \tau_n \geq an, \, D(e) > \tau_n) \,.
\end{aligned} \tag{3.26}$$

Taking expectations gives

$$\begin{aligned}
& \mathbf{Q}(\mathbb{T} \in \mathcal{G}_k) \sum_{\ell=k[2]} \sum_{|x|=k} P^e_{\omega_p}(\tau_k = \ell, X_\ell = x) E_{\mathbf{Q}}[W_n(x,\ell)] \\
(3.27) \qquad & \leq \left(\frac{1+p'}{1+p}\right)^{bn} \mathbb{Q}^e(bn > \tau_n \geq an,\, D(e) > \tau_n)\,.
\end{aligned}$$

As before,

$$\begin{aligned}
& \mathbb{Q}^e(bn > \tau_n \geq an,\, D(e) = \infty) + \mathbb{Q}^e(bn > \tau_n \geq an,\, \infty > D(e) > \tau_n) \\
= \; & \mathbb{Q}^e(bn > \tau_n \geq an,\, D(e) > \tau_n) \\
\geq \; & \mathbb{Q}^e(bn > \tau_n \geq an,\, D(e) = \infty)\,.
\end{aligned}$$

Since $\mathbb{Q}^e(bn > \tau_n \geq an,\, \infty > D(e) > \tau_n) \leq \mathbb{Q}^e(bn > \tau_n \geq an,\, D(e) > \tau_n) E_{\mathbf{Q}}[1-\beta]$, we get

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e(bn > \tau_n \geq an,\, D(e) > \tau_n) = \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e(bn > \tau_n \geq an \,|\, D(e) = \infty)\,.$$

Consequently, we have by (3.24) and (3.27)

$$\limsup_{n\to\infty} \mathbb{Q}^e(bn > \tau_n \geq an) \leq b \ln\left(\frac{1+p'}{1+p}(1+\delta_k)\right) + \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{Q}^e(bn > \tau_n \geq an \,|\, D(e) = \infty)\,.$$

Since $\mathbb{Q}^e(cn > \tau_n > bn) \geq \mathbb{Q}^e(cn > \tau_n > bn,\, D(e) = \infty)$, we prove equation (3.7) by taking p' arbitrarily close to p, and letting k tend to infinity.

We prove now the quenched equality (3.8). For any environment ω, construct the environment $f_p(\omega)$ by setting $A(x) = i$ ($:=$ ess inf A) for any $|x| \leq k$. We construct also for $p' > p$, an environment $f_{p'}(\omega)$ by picking independently $A(x)$ in $[i, p'/\nu_{min}]$ for any $x \leq k$, where $A(x)$ is distributed as A conditioned on $A \in [i, p'/\nu_{min}]$. By equation (3.23), we have almost surely

$$\limsup_{n\to\infty} \frac{1}{n} \ln P^e_\omega(bn > \tau_n \geq an) \leq \limsup_{n\to\infty} \frac{1}{n} P^e_{f_p(\omega)}(bn > \tau_n \geq an, D(e) > \tau_n) + b\ln(1+\delta_k)\,.$$

Equation (3.26) applied to the environment $f_{p'}(\omega)$, together with Theorem 3.2 shows that

$$\limsup_{n\to\infty} \frac{1}{n} \ln P^e_{f_p(\omega)}(bn > \tau_n \geq an, D(e) > \tau_n) \leq -I_q(b) + b\ln\frac{1+p'}{1+p}\,.$$

Let p' tend to p to get that almost surely,

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_{f_p(\omega)}(bn > \tau_n \geq an, D(e) > \tau_n) \leq -I_q(b)\,.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_\omega^e(bn > \tau_n \geq an) \leq -I_q(b) + b\ln(1+\delta_k)\,.$$

When k goes to infinity, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_\omega^e(bn > \tau_n > an) \leq -I_q(b)\,,$$

which gives equation (3.8).

3.4 Proof of Proposition 1.3

Recall that, for any $\theta \in \mathbb{R}$,

$$\psi(\theta) := \ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} \omega(e,e_i)^\theta \right]\right).$$

Obviously, for any $n \in \mathbb{N}$,

$$\frac{1}{n}\ln\left(\mathbb{Q}^e\left(\tau_n = n\right)\right) = \ln\left(E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} \omega(e,e_i) \right]\right) = \psi(1)\,.$$

This proves (1.8). For the quenched case, we have that

$$P_\omega^e\left(\tau_n = n\right) = \sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})\,,$$

where x_k is the ancestor of the vertex x at generation k. We observe that we are reduced to the study of a generalized multiplicative cascade, as studied in [Liu00]. The following lemma is well-known in the case of a regular tree (see [Fra95] and [CV06]). We extend it easily to a Galton–Watson tree.

Lemma 3.6. *We have* $\lim_{n\to\infty} \frac{1}{n} \ln(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k,x_{k+1})) = \inf_{]0,1]} \frac{1}{\theta}\psi(\theta)\,.$

Proof. When $\psi'(1) < \psi(1)$, Biggins [Big77] shows that $\lim_{n\to\infty} \frac{1}{n} \ln(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k,x_{k+1})) = \psi(1) = \inf_{]0,1]} \frac{1}{\theta}\psi(\theta)$. Therefore let us assume that $\psi'(1) \geq \psi(1)$. By the argument of [Fra95], we obtain,

$$\liminf_{n\to\infty} \frac{1}{n} \ln\left(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k,x_{k+1}) \right) \geq \inf_{]0,1]} \frac{1}{\theta}\psi(\theta)\,.$$

Finally, let $\theta \in]0, \theta_c[$ where $\psi(\theta_c) = \inf_{]0,1]} \frac{1}{\theta}\psi(\theta)$. Since $\left(\sum_i a_i\right)^\theta \leq \sum_i a_i^\theta$ for any $(a_i)_i$ with $a_i \geq 0$, it yields that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1}) \right) \leq \frac{1}{\theta} \limsup_{n\to\infty} \frac{1}{n} \ln \left(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})^\theta \right) .$$

We see that (still by [Big77]) $\lim_{n\to\infty} \frac{1}{n} \ln(\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})^\theta) = \psi(\theta)$. It remains to let θ tend to θ_c. $\square$

4 The subexponential regime : Theorem 1.4

We prove (1.10) and (1.11) separately. We recall that the speed v of the walk verifies $v = \frac{E_{\mathbb{S}^e}\left[|X_{\Gamma_1}|\right]}{E_{\mathbb{S}^e}[\Gamma_1]}$.

Proof of Theorem 1.4 : equation (1.10). Suppose that either "$i < \nu_{min}^{-1}$ and $q_1 = 0$" or "$i < \nu_{min}^{-1}$ and $s < 1$" . Let $a > 1/v$ and $c_{24} > 0$ such that $c_{24} < \left(E_{\mathbb{S}^e}\left[X_{\Gamma_1}\right]\right)^{-1}$. We have

$$\mathbb{S}^e\left(\tau_n \geq an\right) \geq \mathbb{S}^e\left(\Gamma_{nc_{24}} \geq an\right) - \mathbb{S}^e\left(\Gamma_{nc_{24}} > \tau_n\right) .$$

The second term on the right-hand side decays exponentially by Cramér's Theorem applied to the random walk $(|X_{\Gamma_n}|, n \geq 0)$ (recall that $|X_{\Gamma_1}|$ has exponential moments by Fact A). The simple inequality $\mathbb{S}^e\left(\Gamma_{nc_{24}} \geq an\right) \geq \mathbb{S}^e\left(\Gamma_1 \geq an\right)$ thus implies by Proposition 2.2 the lower bound of (1.10). Hence, we turn to the upper bound of (1.10). Part (i) of Lemma 6.3 of [DGPZ02] states :

Lemma A (Dembo et al. [DGPZ02]) *Let Y_1, $Y_2, \ldots$ be an i.i.d. sequence with $E(Y_1^2) < \infty$. If $P(Y_1 \geq x) \leq \exp(-cx^\gamma)$ for some $0 < \gamma < 1$, $c > 0$ and all x large enough, then for all $t > E[Y_1]$,*

$$\limsup_{n\to\infty} n^{-\gamma} \ln P\left(\frac{1}{n}\sum_{j=1}^{n} Y_j \geq t\right) \leq -c(t - E[Y_1])^\gamma .$$

By Proposition 2.2, $Y_1 = \Gamma_1$ meets the conditions of the lemma. Therefore, take in lemma A, $Y_i = \Gamma_i - \Gamma_{i-1}$ and $t = a/c_{25}$ where c_{25} is such that

$$\left(E_{\mathbb{S}^e}\left[|X_{\Gamma_1}|\right]\right)^{-1} < c_{25} < a\left(E_{\mathbb{S}^e}\left[\Gamma_1\right]\right)^{-1} .$$

In particular, we have $t > E_{\mathbb{S}^e}[\Gamma_1]$. As a result, $\mathbb{S}^e\left(\Gamma_n > tn\right)$ is stretched exponential. We also know that $\mathbb{S}^e\left(|X_{\Gamma_{nc_{25}}}| \leq n\right)$ is exponentially small by Cramér's Theorem ($1/c_{25}$ <

$E_{\mathbb{S}^e}[|X_{\Gamma_1}|])$. The relation $\mathbb{S}^e(\tau_n \ge an) \le \mathbb{S}^e(\Gamma_{nc_{25}} \ge an) + \mathbb{S}^e\left(|X_{\Gamma_{nc_{25}}}| \le n\right)$ thus completes the proof. □

We finish with the case "$\Lambda < \infty$".

Proof of Theorem 1.4 : equation (1.11). Suppose that $\Lambda < \infty$ and let a, c_{24} and c_{25} be as before. We write

$$\begin{aligned}\mathbb{S}^e(\Gamma_{nc_{24}} \ge an) &\ge \sum_{k=1}^{nc_{24}} \mathbb{S}^e(\{\Gamma_k - \Gamma_{k-1} \ge an\} \cap \{\Gamma_\ell - \Gamma_{\ell-1} < an, \forall \ell \ne k\}) \\ &= nc_{24}\mathbb{S}^e(\Gamma_1 \ge an)\,\mathbb{S}^e(\Gamma_1 < an)^{nc_{24}-1}.\end{aligned}$$

By Proposition 2.4, $\mathbb{S}^e(\Gamma_1 \ge an) = n^{-\Lambda+o(1)}$. Therefore $\mathbb{S}^e(\Gamma_1 < an)^{nc_{24}-1}$ tends to 1 (since $\Lambda > 1$). Consequently,

$$\mathbb{S}^e(\Gamma_{nc_{24}} \ge an) \ge n^{1-\Lambda+o(1)},$$

which gives the lower bound of (1.11), by the inequality $\mathbb{S}^e(\tau_n \ge an) \ge \mathbb{S}^e(\Gamma_{nc_{24}} \ge an) - \mathbb{S}^e(\Gamma_{nc_{24}} > \tau_n)$. Turning, to the upper bound, write as before $\mathbb{S}^e(\tau_n \ge an) \le \mathbb{S}^e(\Gamma_{nc_{25}} \ge an) + \mathbb{S}^e\left(|X_{\Gamma_{nc_{25}}}| \le n\right)$. We already know that $\mathbb{S}^e\left(|X_{\Gamma_{nc_{25}}}| \le n\right)$ is exponentially small. Let $H_n := \Gamma_n - E_{\mathbb{S}^e}[\Gamma_1]n$. When $E[H_1^p] < \infty$, example 2.6.5 of [Pet75] says that if $p \ge 2$,

$$P(H_n > x) \le (1 + 2/p)^p\, nE[H_1^p]x^{-p} + \exp(-2(p+2)^{-2}e^{-p}x^2/(nE[H_1^2]))$$

and example 2.6.20 of [Pet75], combined with Chebyshev's inequality, shows that if $1 \le p \le 2$,

$$P(H_n > x) \le (2 - 1/n)\, nE[H_1^p]x^{-p}.$$

By Proposition 2.4, $E[H_1^p] < \infty$, for any $p < \Lambda$. We take $x = (\frac{a}{c_{25}}E_{\mathbb{S}^e}[|X_{\Gamma_1}|] - E_{\mathbb{S}^e}[\Gamma_1])n$ to see that $\mathbb{S}^e(\Gamma_{nc_{25}} \ge an) \le c(p)n^{1-p}$ for any $p < \Lambda$. Let p tend to Λ in order to complete the proof of equation (1.11). □

5 Bound of the speed

In this section, we deal with bounds on the speed v of the RWRE. We suppose that $\Lambda > 1$, thus the speed is positive. The link with the previous work lies in the use of the moments of regeneration epochs, which allows us to prove the following lemma.

Lemma 5.1. *Define for any $n \geq 0$, the (random) integer $u(n)$ such that $\Gamma_{u(n)} \leq \tau_n < \Gamma_{u(n)+1}$. If $\Lambda > 1$, we have*

$$\lim_{n\to\infty} \frac{1}{n} E^e_{\mathbb{S}}[\Gamma_{u(n)}] = \frac{1}{v}\,.$$

Proof of Lemma 5.1. We recall that $v = \frac{E^e_{\mathbb{S}}[|X_{\Gamma_1}|]}{E^e_{\mathbb{S}}[\Gamma_1]}$. Let $\eta > \frac{1}{E^e_{\mathbb{S}}[|X_{\Gamma_1}|]}$. We observe that

$$\begin{aligned} E^e_{\mathbb{S}}[\Gamma_{u(n)}] &\leq E^e_{\mathbb{S}}[\Gamma_{u(n)} \mathbb{1}_{\{\Gamma_{\eta n} < \tau_n\}}] + E^e_{\mathbb{S}}[\Gamma_{\eta n}] \\ &\leq E^e_{\mathbb{S}}[\Gamma_n \mathbb{1}_{\{\Gamma_{\eta n} < \tau_n\}}] + \eta n E^e_{\mathbb{S}}[\Gamma_1]\,. \end{aligned}$$

Let $p \in (1, \Lambda)$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. By Proposition 2.4, we know that $E^e_{\mathbb{S}}[\Gamma_1^p] < \infty$. By Cauchy-Schwartz inequality,

$$E^e_{\mathbb{S}}[\Gamma_n \mathbb{1}_{\{\Gamma_{\eta n} < \tau_n\}}] \leq E^e_{\mathbb{S}}[\Gamma_n^p]^{1/p} \mathbb{S}^e(\Gamma_{\eta n} < \tau_n)^{1/q} \leq n^{1+\frac{1}{p}} E^e_{\mathbb{S}}[\Gamma_1^p]^{1/p} \mathbb{S}^e(\Gamma_{\eta n} < \tau_n)^{1/q}$$

which is exponentially small by Cramér's Theorem ($|X_{\Gamma_1}|$ has exponential moments by Fact A). Therefore $\limsup_{n\to\infty} \frac{1}{n} E^e_{\mathbb{S}}[\Gamma_{u(n)}] \leq \eta E^e_{\mathbb{S}}[\Gamma_1]$, which gives by letting η tend to $\frac{1}{E^e_{\mathbb{S}}[|X_{\Gamma_1}|]}$,

$$\limsup_{n\to\infty} \frac{1}{n} E^e_{\mathbb{S}}[\Gamma_{u(n)}] \leq \frac{1}{v}\,.$$

Similarly, for $\eta < \frac{1}{E^e_{\mathbb{S}}[|X_{\Gamma_1}|]}$,

$$E^e_{\mathbb{S}}[\Gamma_{u(n)}] \geq E^e_{\mathbb{S}}[\Gamma_{\eta n}] - E^e_{\mathbb{S}}[\Gamma_{\eta n} \mathbb{1}_{\{\Gamma_{\eta n} \geq \tau_n\}}]\,.$$

We proceed as before to see that

$$\liminf_{n\to\infty} \frac{1}{n} E^e_{\mathbb{S}}[\Gamma_{u(n)}] \geq \frac{1}{v}\,,$$

which completes the proof. □

Let for any vertex x and generation n,

$$\begin{aligned} N_x(T_{\overleftarrow{e}}) &:= \sum_{k=0}^{T_{\overleftarrow{e}}} \mathbb{1}_{\{X_k = x\}}\,, \\ N_n(T_{\overleftarrow{e}}) &:= \sum_{|y|=n} N_y(T_{\overleftarrow{e}})\,, \end{aligned}$$

which stand for the time spent in x and in generation n before reaching $\overleftarrow{e}$. We have already defined

$$\begin{aligned} N_x &:= \sum_{k\geq 0} \mathbb{1}_{\{X_k = x\}}\,, \\ N_n &= \sum_{|y|=n} N_y\,. \end{aligned}$$

Lemma 5.2. *The following equation holds :*

$$\lim_{n\to\infty}\frac{1}{n}E^e_{\mathbb{Q}}\left[\sum_{k=0}^{n}N_k(T_{\overleftarrow{e}})\right]=\frac{\mathbb{Q}^e(D(e)=\infty)}{v}\,.$$

Proof. We observe that

$$0\le\frac{1}{n}E^e_{\mathbb{S}}\left[\sum_{k=0}^{n}N_k\right]-\frac{1}{n}E^e_{\mathbb{S}}\left[\Gamma_{u(n)}\right]\le\frac{1}{n}E^e_{\mathbb{S}}\left[\Gamma_{u(n)+1}-\Gamma_{u(n)}\right]\le\frac{1}{n}E^e_{\mathbb{S}}\left[M_n\right]\,,$$

where $M_n:=\max_{1\le k\le n}(\Gamma_k-\Gamma_{k-1})$. We show that the last quantity converges towards 0

$$\begin{aligned}\frac{1}{n}E^e_{\mathbb{S}}\left[M_n\right] &\le \frac{1}{\sqrt{n}}+\frac{1}{n}E^e_{\mathbb{S}}\left[M_n\mathbb{1}_{\{M_n\ge\sqrt{n}\}}\right]\\ &\le \frac{1}{\sqrt{n}}+\frac{1}{n}\sum_{k=1}^{n}E^e_{\mathbb{S}}\left[(\Gamma_k-\Gamma_{k-1})\mathbb{1}_{\{\Gamma_k-\Gamma_{k-1}\ge\sqrt{n}\}}\mathbb{1}_{\{M_n=\Gamma_k-\Gamma_{k-1}\}}\right]\\ &\le \frac{1}{\sqrt{n}}+E^e_{\mathbb{S}}\left[\Gamma_1\mathbb{1}_{\{\Gamma_1\ge\sqrt{n}\}}\right]\,,\end{aligned}$$

which tends to 0. By Lemma 5.1, it yields that

$$\lim_{n\to\infty}\frac{1}{n}E^e_{\mathbb{Q}}\left[\sum_{k=0}^{n}N_k\,\middle|\,D(e)=\infty\right]=\frac{1}{v}\,.$$

Finally,

$$0\le E^e_{\mathbb{Q}}\left[\sum_{k=0}^{n}N_k(T_{\overleftarrow{e}})\right]-E^e_{\mathbb{Q}}\left[\sum_{k=0}^{n}N_k\mathbb{1}_{\{T_{\overleftarrow{e}}=\infty\}}\right]\le E^e_{\mathbb{Q}}\left[T_{\overleftarrow{e}}\mathbb{1}_{\{T_{\overleftarrow{e}}<\infty\}}\right]\tag{5.1}$$

Let us see why the last expectation is finite. Let E be the following event : the root has at least two children, the walk goes to its first child e_1, comes back to the root, then goes to the second child e_2 and never comes back. On the event E, the first regeneration time is greater than the time the walk takes from e_1 to go back to the root. We deduce that

$$\begin{aligned}\infty>E^e_{\mathbb{Q}}\left[\Gamma_1\mathbb{1}_{\{D(e)=\infty\}}\right] &\ge E^e_{\mathbb{Q}}\left[\mathbb{1}_E\Gamma_1\mathbb{1}_{\{D(e)=\infty\}}\right]\\ &\ge E_{\mathbf{Q}}\left[\mathbb{1}_{\{\nu(e)\ge2\}}\omega(e,e_1)E^{e_1}_{\omega}\left[T_e\mathbb{1}_{\{T_e<\infty\}}\right]\omega(e,e_2)\beta(e_2)\right]\\ &= E_{\mathbf{Q}}\left[\mathbb{1}_{\{\nu(e)\ge2\}}\omega(e,e_1)\omega(e,e_2)\beta(e_2)\right]E^e_{\mathbb{Q}}\left[T_{\overleftarrow{e}}\mathbb{1}_{\{T_{\overleftarrow{e}}<\infty\}}\right]\,.\end{aligned}$$

Therefore, equation (5.1) tells that

$$\lim_{n\to\infty}\frac{1}{n}E^e_{\mathbb{Q}}\left[\sum_{k=0}^{n}N_k(T_{\overleftarrow{e}})\right]=\frac{\mathbb{Q}^e(D(e)=\infty)}{v}\,.\ \square$$

We can now prove

Lemma 5.3. *We have*

$$\frac{2}{E_{\mathbf{Q}}[\beta]} - 1 \le \frac{1}{v} \le E_{\mathbf{Q}}\left[\frac{2}{\beta}\right] - 1 \,.$$

Proof. We define N_x^* as the number of times the walk goes from $\overleftarrow{x}$ to x, and we define similarly $N_x^*(T_{\overleftarrow{e}})$, N_n^*, $N_n^*(T_{\overleftarrow{e}})$. We observe that

$$N_k(T_{\overleftarrow{e}}) = N_k^*(T_{\overleftarrow{e}}) + N_{k+1}^*(T_{\overleftarrow{e}}) - \mathbb{1}_{\{T_{\overleftarrow{e}}=\infty\}} \,. \tag{5.2}$$

Call R the unique ray such that $R := \{x \in \mathbb{T} : \exists N\, \forall n \ge N, X_n \text{ is a descendant of } x\}$. By the Markov property,

$$P_\omega^x(x \in R, T_{\overleftarrow{e}} = \infty) = \beta(x) + (1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})P_\omega^x(x \in R, T_{\overleftarrow{e}} = \infty)\,.$$

Therefore,

$$P_\omega^x(x \in R, T_{\overleftarrow{e}} = \infty) = \frac{\beta(x)}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})} = \left(1 + E_\omega^x\left[N_x^*(T_{\overleftarrow{e}})\right]\right)\beta(x)\,.$$

On the other hand,

$$\begin{aligned} E_{\mathbb{Q}}^e\left[N_n^*(T_{\overleftarrow{e}})\right] &= E_{\mathbb{Q}}^e\left[\sum_{|x|=n} N_x^*(T_{\overleftarrow{e}})\right] \\ &= E_{\mathbf{Q}}\left[\sum_{|x|=n} \frac{P_\omega^e(T_x < T_{\overleftarrow{e}})}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})}\right]. \end{aligned}$$

We observe that $E\left[\frac{aX}{b+cX}\right] \le \frac{aE[X]}{b+cE[X]}$ for any constants $a,b,c>0$. It yields the following lines

$$\begin{aligned} E\left[\frac{aX}{b+cX}\right](b+cE[X]) &\le aE[X]\,, \\ bE\left[\frac{aX}{b+cX}\right] &\le \left(a - cE\left[\frac{aX}{b+cX}\right]\right)E[X] \\ &= E\left[\frac{ab}{b+cX}\right]E[X]\,, \\ E\left[\frac{aX}{b+cX}\right] &\le E\left[\frac{a}{b+cX}\right]E[X]\,. \end{aligned}$$

We notice that $\beta(x)$ is independent of the first n generations. Therefore, we can take by conditioning,

$$X = \frac{1}{\beta(x)}\,,$$

$$\begin{aligned} a &= P_\omega^e(T_x < T_{\overleftarrow{e}})\,, \\ b &= P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})\,, \\ c &= 1 - P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})\,. \end{aligned}$$

It gives that, if F_n stands for the environment of the first n generations,

$$\begin{aligned} & E_{\mathbf{Q}}\left[\sum_{|x|=n} \frac{P_\omega^e(T_x < T_{\overleftarrow{e}})}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})} \,\Big|\, F_n\right] \\ = & \sum_{|x|=n} E_{\mathbf{Q}}\left[\frac{P_\omega^e(T_x < T_{\overleftarrow{e}})}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})} \,\Big|\, F_n\right] \\ \leq & \sum_{|x|=n} E_{\mathbf{Q}}\left[\frac{P_\omega^e(T_x < T_{\overleftarrow{e}})\beta(x)}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})} \,\Big|\, F_n\right] E_{\mathbf{Q}}\left[\frac{1}{\beta}\right] \\ = & \sum_{|x|=n} E_{\mathbf{Q}}\left[P_\omega^e(x \in R, T_{\overleftarrow{e}} = \infty) \,|\, F_n\right] E_{\mathbf{Q}}\left[\frac{1}{\beta}\right] \\ = & \; E_{\mathbf{Q}}\left[P_\omega^e(T_{\overleftarrow{e}} = \infty) \,|\, F_n\right] E_{\mathbf{Q}}\left[\frac{1}{\beta}\right]\,. \end{aligned}$$

Consequently,

$$E_{\mathbb{Q}}^e\left[N_n^*(T_{\overleftarrow{e}})\right] \leq E_{\mathbf{Q}}\left[\frac{1}{\beta}\right] \mathbb{Q}^e(T_{\overleftarrow{e}} = \infty)\,.$$

Similarly, take now

$$\begin{aligned} X &= \beta(x)\,, \\ a &= P_\omega^e(T_x < T_{\overleftarrow{e}})\,, \\ b &= 1 - P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})\,, \\ c &= P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})\,. \end{aligned}$$

It yields

$$\begin{aligned} E_{\mathbf{Q}}\left[\frac{P_\omega^e(T_x < T_{\overleftarrow{e}})}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})} \,\Big|\, F_n\right] & \geq E_{\mathbf{Q}}\left[\frac{P_\omega^e(T_x < T_{\overleftarrow{e}})\beta(x)}{1-(1-\beta(x))P_\omega^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}})} \,\Big|\, F_n\right] \frac{1}{E_{\mathbf{Q}}[\beta]} \\ & = E_{\mathbf{Q}}\left[P_\omega^e(x \in R, T_{\overleftarrow{e}} = \infty) \,|\, F_n\right] \frac{1}{E_{\mathbf{Q}}[\beta]}\,, \end{aligned}$$

which implies that $E_{\mathbb{Q}}^e\left[N_n^*(T_{\overleftarrow{e}})\right] \geq \mathbb{Q}^e(T_{\overleftarrow{e}} = \infty)\frac{1}{E_{\mathbf{Q}}[\beta]}$. It follows by (5.2)

$$\frac{2\mathbb{Q}^e(T_{\overleftarrow{e}} = \infty)}{E_{\mathbf{Q}}[\beta]} - \mathbb{Q}^e(T_{\overleftarrow{e}} = \infty) \leq E_{\mathbb{Q}}^e[N_k(T_{\overleftarrow{e}})] \leq 2\mathbb{Q}^e(T_{\overleftarrow{e}} = \infty)E_{\mathbf{Q}}\left[\frac{1}{\beta}\right] - \mathbb{Q}^e(T_{\overleftarrow{e}} = \infty)\,.$$

Lemma 5.2 completes the proof. □

We find as a corollary the following bounds for the speed.

Corollary 5.4. *If* $\Lambda > 1$, *we have*

$$\frac{E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right]-1}{E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right]+1} \geq v \geq \frac{1-E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right]}{1+E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right]}$$

Proof of Corollary 5.4. We know that (see e.g. equation (4.2) of [HS07b])

$$\beta(e)=\frac{\sum_{i=1}^{\nu(e)} A(e_i)\beta(e_i)}{1+\sum_{i=1}^{\nu(e)} A(e_i)\beta(e_i)}.$$

By Jensen's inequality we get,

$$E_{\mathbf{Q}}[\beta] \leq \frac{E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right] E_{\mathbf{Q}}[\beta]}{1+E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right] E_{\mathbf{Q}}[\beta]},$$

which yields that $E_{\mathbf{Q}}[\beta] \leq 1-\frac{1}{E_{\mathbf{Q}}\left[\sum_{i=1}^{\nu(e)} A(e_i)\right]}$. The first inequality comes then from Lemma 5.3. For the lower bound, define

$$\beta^n(x) := P_\omega^x(\tau_n < T_{\overleftarrow{x}})$$

We have, still by equation (4.2) of [HS07b],

$$\begin{aligned}\frac{1}{\beta^n(e)} &= 1+\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)\beta^n(e_i)}\\ &\leq 1+\frac{1}{\left(\sum_{i=1}^{\nu(e)} A(e_i)\right)^2}\sum_{i=1}^{\nu(e)}\frac{A(e_i)}{\beta^n(e_i)}.\end{aligned}$$

Taking the expectation yields

$$E_{\mathbf{Q}}[\frac{1}{\beta^n(e)}] \leq 1+E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right] E_{\mathbf{Q}}\left[\frac{1}{\beta^{n-1}(e)}\right].$$

If $E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu} A(e_i)}\right] < 1$, we get by recurrence

$$E_{\mathbf{Q}}\left[\frac{1}{\beta^n(e)}\right] \leq \frac{1-E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right]^n}{1-E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu(e)} A(e_i)}\right]}.$$

Finally, use the monotone convergence to see that

$$E_{\mathbf{Q}}\left[\frac{1}{\beta}\right] \leq \frac{1}{1 - E_{\mathbf{Q}}\left[\frac{1}{\sum_{i=1}^{\nu} A(e_i)}\right]}.$$

Lemma 5.3 completes the proof. □

Acknowledgements : I am very grateful to Zhan Shi for his guidance in the redaction of this work. I thank also the anonymous referee for suggesting me several improvements.

Deuxième partie

Marches aléatoires branchantes

Chapitre IV

Survival of branching random walks with absorption [1]

Summary. We consider a branching random walk on $\mathbb{R}$ starting from $x > 0$ and with a killing barrier at 0. At each step, particles give birth to b children, which move independently. Particles that enter the negative half-line are killed. Biggins et al. [BLSW91] characterized the domain of almost sure extinction. In this case, we find accurate asymptotics for the survival probabilities at time n, when n tends to infinity.

Key words. Branching random walk, survival probability.

AMS subject classifications. 60J80.

1 Introduction

We consider a branching random walk on $\mathbb{R}$ with an absorbing barrier at the origin. At time n, each individual of the surviving population gives birth to a fixed number of children, which move independently from the position of their father. Particles that enter the negative half-line are immediately killed, and do not have any descent.

Precisely, take $b \in \mathbb{N}^*$. Let $\mathcal{T}$ be a rooted b-ary tree, with the partial order $v < u$ if v is an ancestor of u, and let $|u|$ denote the generation of u, the generation of the root being zero. We attach i.i.d random variables $(X_u,\, u \in \mathcal{T}, |u| \geq 1)$ (X will denote a generic random

1. This chapter is joint work with Bruno Jaffuel.

variable with the common distribution). For $u \in \mathcal{T}$, we define the position $S(u)$ of u by :

$$S(u) = x + \sum_{v<u} X_v \,,$$

where x is the position of the ancestor (the root). The surviving population Z_n at time n is the number of particles that never touched the negative half-line :

$$Z_n := \#\{|u| = n \,:\, S(v) > 0 \,\forall\, v \leq u\}.$$

Any individual u such that $S(u) \leq 0$ dies, and then has no children. Therefore we are only interested in individuals which have all their ancestors, including themselves, at the right of the barrier 0.

Kesten [Kes78], and recently Harris and Harris [HH07] worked on the branching Brownian motion with absorption, which is the continuous analog of our problem. We refer to Derrida [DS07], [SD08] for a more physical point of view on killed branching random walks. In the setting of branching Brownian motion (without absorption), large deviations probabilities on the speed of the rightmost particle was given by Chauvin and Rouault [CR88].

The first natural question that arises is whether the population ultimately dies. The frontier between extinction and non extinction is given by Biggins et al [BLSW91]. We introduce

$$\phi(t) := E[e^{tX}]\,, \tag{1.1}$$

$$\gamma := \inf_{t\geq 0} E\left[e^{tX}\right]. \tag{1.2}$$

We suppose that

- there exist $s_1 > 0$ and $s_2 > 0$ such that $E[e^{\theta X}] < \infty$ for $-s_1 < \theta < s_2$.
- ϕ reaches its infimum (say at $t = \nu$) and $-s_1 < \nu < s_2$.
- the distribution of X is non-lattice.

We have

(i) If $\gamma \leq 1/b$, there is almost sure extinction.

(ii) If $\gamma > 1/b$, the process survives with positive probability.

Throughout this paper, we focus on the extinction case (i). We assume that $\gamma \leq 1/b$ where we necessarily have $\phi'(0) = E[X] \leq 0$. Particles are attracted to the barrier, and strongly enough to compensate the reproduction. Define

$$u_n(x) := P^x(Z_n > 0)$$

which is the probability for the process to survive until generation n, starting from x. We already know that this probability tends to zero. The aim of this paper is to estimate the rate of decay of u_n. Our first theorem deals with the subcritical case. Let $X_1, X_2, \dots$ be i.i.d random variables distributed as X, and define $S_n := S_0 + \sum_{k=1}^n X_k$. Under the probability P^z, $S_0 = z$ almost surely. We introduce $I_k := \inf\{S_j,\, j \le k\}$ and for any $x \ge 0$

$$\widetilde{V}(x) := 1 + \sum_{k=1}^{\infty} \gamma^{-k} E^0[e^{\nu S_k} \mathbb{1}_{\{S_k = I_k \ge -x\}}] \,. \tag{1.3}$$

Theorem 1.1. *If $\gamma < 1/b$, then there exists a constant $C_1 > 0$ independent of x such that*

$$u_n(x) \sim C_1 e^{\nu x} \widetilde{V}(x) b^n \gamma^n n^{-3/2}$$

for any x point of continuity of $\widetilde{V}(x)$.

The proof makes use of the following result for one-dimensional random walks taken from Doney [Don89].

Theorem A (Doney [Don89]) *There exists a constant $C_2 > 0$ independent of x such that*

$$P^x(I_n > 0) \sim C_2 e^{\nu x} \widetilde{V}(x) n^{-3/2} \gamma^n \tag{1.4}$$

for any x point of continuity of $\widetilde{V}$.

Consequently, the mean population at time n is given by $E^x[Z_n] = C_2 e^{\nu x} \widetilde{V}(x) n^{-3/2} b^n \gamma^n$. In light of Theorem 1.1, we can therefore state that

$$E^x[Z_n \,|\, Z_n > 0] \to K \,.$$

Conditionally on non-extinction, the mean population converges to a constant independent of the starting point. Our next result concerns the critical case $\gamma = 1/b$. We find here that the probability to survive is of order smaller than $E^x[Z_n]$, which is in contrast with the subcritical case.

Theorem 1.2. *Suppose that* $\gamma := E[e^{\nu X}] = \frac{1}{b}$*. Let* $\sigma^2 := \phi''(\nu)/\phi(\nu)$*. We have, for any* $x > 0$*,*

$$\ln(P^x(Z_n > 0)) \sim -\left(\frac{3\sigma^2\nu^2\pi^2}{2}\right)^{1/3} n^{1/3}.$$

The paper is organized as follows. Section 2 presents a recurrence formula for the surviving probability. Section 3 contains the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2.

2 Decomposition of the branching random walk

Recall that $S_n = S_0 + X_1 + X_2 + \ldots + X_n$ is the one-dimensional random walk associated to the branching process. Define

$$\tau_0 := \inf\{k \geq 1 \,:\, S_k \leq 0\}.$$

the first passage time to 0. Since $E[X_1] < 0$, we know that S_n drifts to $-\infty$ and $\tau_0 < \infty$ almost surely. Notice also that for any $|u| = n$, $S(u)$ is distributed as S_n.

Let for $h \in]0,1]$:

$$B(h) := \frac{1-(1-h)^b}{bh}.$$

Our first lemma shows a recurrence formula for $u_n(x) := P^x(Z_n > 0)$.

Lemma 2.1. *For any* $x \in \mathbb{R}$ *and* $n \geq 1$*, we have*

$$u_n(x) = b\mathbb{1}_{\{x>0\}} E\left[u_{n-1}(x+X_1)\right] B\left(E\left[u_{n-1}(x+X_1)\right]\right).$$

Proof. Firstly, we obviously have $u_n(x) = 0$ if $x \leq 0$. Therefore take $x > 0$. We write that the process survives (until the n^{th} generation) if and only if at least one of the individuals in the first generation has a descendant in the n^{th} generation.

$$u_n(x) = E\left[\left(1 - \prod_{i=1}^{b}(1-\mathbb{1}_{A_i})\right)\right],$$

with A_i the event $\{$the i^{th} individual in the first generation has a descendant in the n^{th} generation$\}$. Using the branching property of the process, one gets that the events $\{(A_i, i =$

$1, \ldots, b)\}$ are independent and have the same probability equal to $E[u_{n-1}(x+X_1)]$. Put this in the preceding equation to obtain

$$u_n(x) = E\left[1 - (1 - E[u_{n-1}(x+X_1)])^b\right].$$

The conclusion follows from the definition of B. □

Define for any $n \geq 1$ and $x \in \mathbb{R}$:

$$w_n(x) := B\left(E[u_{n-1}(x+X_1)]\right).$$

For future use, notice that

$$\begin{aligned} 1 - w_n(x) &\leq \frac{b-1}{2} E[u_{n-1}(x+X_1)] \\ &\leq \frac{b-1}{2} u_n(x) =: c_3 u_n(x). \end{aligned}$$

This allows us to rewrite the lemma as follows :

$$u_n(x) = \mathbb{1}_{\{x>0\}} E[u_{n-1}(x+X_1)]\, w_n(x). \tag{2.1}$$

Lemma 2.2. *For any $n \geq 0$ and $x \in \mathbb{R}$, we have*

$$u_n(x) = \mathbb{1}_{\{x>0\}} b^n E\left[\mathbb{1}_{\{\tau_0>n\}} \prod_{k=1}^{n} w_k(x+S_{n-k})\right] \tag{2.2}$$

Proof. We proceed by induction. The case $n=0$ is easy since $u_0(x) = \mathbb{1}_{\{x>0\}}$. Now suppose $n \geq 1$ and $x > 0$. By equation (2.1), we have

$$u_n(x) = bE[u_{n-1}(x+X_1)]\, w_n(x). \tag{2.3}$$

Applying the recurrence hypothesis to $u_{n-1}(x+X_1)$ gives :

$$\begin{aligned} u_n(x) &= b^n E\left[\mathbb{1}_{\{x+X_1>0\}} E\left[\mathbb{1}_{\{\tau_0'>(n-1)\}} \prod_{k=1}^{n-1} w_k(x+X_1+S'_{n-k-1})\right]\right] w_n(x) \\ &= b^n E\left[\mathbb{1}_{\{\tau_0>n\}} \prod_{k=1}^{n} w_k(x+S_{n-k})\right] \end{aligned}$$

which completes the proof. □

3 The subcritical case

Let x be a point of continuity of $\widetilde{V}$. Lemma 2.2 together with equation (1.4) imply that Theorem 1.1 is a consequence of the following proposition.

Proposition 3.1. *If x is a point of continuity of $\widetilde{V}$, then*

$$E^x\left[\prod_{k=1}^{n} w_k(S_{n-k})\,\middle|\,\tau_0 > n\right] \text{ converges to a constant as } n \to \infty. \tag{3.1}$$

Furthermore, the limit does not depend on the value of x.

The rest of the section is devoted to the proof of Proposition 3.1. The basic idea of the proof goes back to Harris and Harris [HH07], but several important ingredients (such as stochastic calculus and path decomposition for Bessel bridges) are no longer available in the discrete setting. We prove the lower limit and the upper limit in two distinct subsections. Before, we prove the following lemma.

Lemma 3.2. *Conditionally on $\tau_0 > n$, the random variable $x + X_1 + \ldots + X_n$ converges in law to some S^*. The limiting distribution does not depend on x.*

Remark. This result was originally proved by Iglehart [Igl74] in the case $x = 0$. We have, for any $\lambda \geq 0$,

$$E^0[e^{-\lambda S_n} \mid \tau_0 > n] \to E[e^{-\lambda S^*}]\,. \tag{3.2}$$

We prove that the limit distribution is the same in the case $x > 0$.

Proof of Lemma 3.2. Let $\lambda > 0$, and P_λ the distribution defined by

$$dP_\lambda|_{\mathcal{F}_n} := \frac{e^{-\lambda(S_n - S_0)}}{\phi(-\lambda)^n} dP|_{\mathcal{F}_n}$$

where $\mathcal{F}_n$ is the σ-algebra generated by $(S_k,\, k \leq n)$. We suppose that λ is small enough to have $\phi(-\lambda) < \infty$, that is $\lambda < s_1$. For any $x > 0$, we notice that

$$E^x[e^{-\lambda S_n} \mathbb{1}_{\{\tau_0 > n\}}] = e^{-\lambda x} \phi(-\lambda)^n P_\lambda^x(\tau_0 > n)\,.$$

By Theorem II of [Don89], $P_\lambda^x(\tau_0 > n)$ is equivalent to $e^{(\nu+\lambda)x}\widetilde{V}_\lambda(x) P_\lambda^0(\tau_0 > n)$ with

$$\widetilde{V}_\lambda(x) := 1 + \sum_{k=1}^{\infty} E_{P_\lambda}[e^{(\nu+\lambda)X_1}]^{-k} E_{P_\lambda}^0[e^{(\nu+\lambda)S_k} \mathbb{1}_{\{I_k = S_k > -x\}}]\,,$$

and if x is a continuity point of $\widetilde{V}_\lambda$. This is direct when observing that $\widetilde{V}_\lambda(x)$ is also equal to

$$\widetilde{V}(x) := 1 + \sum_{k=1}^{\infty} \phi(\nu)^{-k} E^0[e^{\nu S_k} \mathbb{1}_{\{I_k = S_k > -x\}}] \,.$$

Therefore, we deduce that when n tends to infinity

$$\begin{aligned} E^x[e^{-\lambda S_n} \mathbb{1}_{\{\tau_0 > n\}}] &\sim e^{\nu x} \phi(-\lambda)^n \widetilde{V}(x) P^0_\lambda(\tau_0 > n) \\ &\sim e^{\nu x} \widetilde{V}(x) E^0[e^{-\lambda S_n} \mathbb{1}_{\{\tau_0 > n\}}] \,. \end{aligned}$$

Since $P^x(\tau_0 > n) \sim e^{\nu x} \widetilde{V}(x) P^0(\tau_0 > n)$, we obtain

$$E^x[e^{-\lambda S_n} \,|\, \tau_0 > n] \sim E^0[e^{-\lambda S_n} \,|\, \tau_0 > n] \,.$$

We conclude by equation (3.2). $\square$

3.1 The upper bound

We prove a simpler version of equation (3.1).

Lemma 3.3. *Let $K \in \mathbb{N}^*$. Then*

$$E^x \left[\prod_{k=1}^{K} w_k(S_{n-k}) \middle| \tau_0 > n \right] \text{ converges to a constant } a_K \text{ as } n \to \infty. \tag{3.3}$$

Proof. Let $\widetilde{S}_i := S_{n-K+i} - S_{n-K}$. We have

$$E^x \left[\mathbb{1}_{\{\tau_0 > n\}} \prod_{k=1}^{K} w_k(S_{n-k}) \right] = E^x \left[\mathbb{1}_{\{\tau_0 > n\}} \prod_{k=0}^{K-1} w_{K-k}(S_{n-K} + \widetilde{S}_k) \right]$$

Since $\widetilde{S}$ is independent of S_{n-K}, we can write

$$E^x \left[\mathbb{1}_{\{\tau_0 > n\}} \prod_{k=0}^{K-1} w_{K-k}(S_{n-K} + \widetilde{S}_k) \right] = E^x \left[\mathbb{1}_{\{\tau_0 > n-K\}} f_K(S_{n-K}) \right]$$

with $f_K(z) := E^z \left[\mathbb{1}_{\{\tau_0 > K\}} \prod_{k=0}^{K-1} w_{K-k}(S_k) \right]$. By Lemma 3.2, we know that S_n, conditioned on $\tau_0 > n$, converges in distribution to S^*. Then by the continuous mapping theorem, $E\left[f_K(S_{n-K}) \,|\, \tau_0 > n-K\right]$ converges to $E\left[f_K(S^*)\right]$. We are allowed to make use of the continuous mapping theorem because S^* has a density (see [Igl74]) and f_K has at most countably many points of discontinuity (indeed, these are the points from which the random walk has a positive probability to reach the origin in at most K steps and stay positive before. They are related to the atoms of the law of X which are at most countably many). Then,

$$E^x\left[\prod_{k=1}^{K} w_k(S_{n-k})\,\Big|\,\tau_0>n\right]=\frac{P^x(\tau_0>n-K)}{P^x(\tau_0>n)}E^x\left[f_K(S_{n-K})\,|\,\tau_0>n-K\right],$$

which tends to $a_K:=\gamma^{-K}E\left[f_K(S^*)\right]$ by equation (1.4). □

We deduce the following upper bound.

Corollary 3.4. *We have*

$$\limsup_{n\to\infty} E^x\left[\prod_{k=0}^{n}(1-w_k(S_{n-k}))\,\Big|\,\tau_0>n\right]\le\inf_{K\ge1}a_K\,.$$

Proof. We observe that for any $K\geq 1$,

$$E^x\left[\prod_{k=1}^{n} w_k(S_{n-k})\,\Big|\,\tau_0>n\right]\le E^x\left[\prod_{k=1}^{K} w_k(S_{n-k})\,\Big|\,\tau_0>n\right].$$

The corollary follows from Lemma 3.3, by taking the infimum on K. □

We next show that the upper bound is also a lower bound, hence proving Proposition 3.1.

3.2 The lower bound

We first show the following lemma.

Lemma 3.5. *For any $\varepsilon>0$, there exists $K=K(\varepsilon)$, such that for n large enough*

$$E^x\left[\prod_{k=0}^{n} w_k(S_{n-k})\,\Big|\,\tau_0>n\right]\ge(1-\varepsilon)E^x\left[\prod_{k=0}^{K} w_k(S_{n-k})\,\Big|\,\tau_0>n\right]-\varepsilon\,.$$

Proof. We have, for any $K\leq n$,

$$\begin{aligned}&E^x\left[\prod_{k=0}^{n} w_k(S_{n-k})\,\Big|\,\tau_0>n\right]\\ \geq\;&(1-\varepsilon)E^x\left[\prod_{k=0}^{K} w_k(S_{n-k})\,\Big|\,\tau_0>n\right]-P^x\left(\prod_{k=K}^{n} w_k(S_{n-k})<1-\varepsilon\,\Big|\,\tau_0>n\right).\end{aligned}$$

Therefore, we need to show that the term

$$P^x\left(\prod_{k=K}^{n} w_k(S_{n-k}) < 1-\varepsilon \,\middle|\, \tau_0 > n\right)$$

is small when K is large. We split it into three parts, by observing that

$$\begin{aligned} & P^x\left(\prod_{k=K}^{n} w_k(S_{n-k}) < 1-\varepsilon \,\middle|\, \tau_0 > n\right) \\ \leq\ & P^x\left(S_n \geq M \,\middle|\, \tau_0 > n\right) \\ & + P^x\left(\exists\, K \leq k < n \text{ such that } S_n - S_{n-k} \leq -k^{2/3},\, S_n \leq M \,\middle|\, \tau_0 > n\right) \\ & + P^x\left(\prod_{k=K}^{n} w_k(S_{n-k}) < 1-\varepsilon,\, S_{n-k} \leq M + k^{2/3},\, K \leq k \leq n \,\middle|\, \tau_0 > n\right) . \end{aligned}$$

By Lemma 3.2, there exists M such that $P^x\left(S_n \geq M \,\middle|\, \tau_0 > n\right) < \varepsilon/2$ for n large enough. Let us consider now the third term of the right-hand side. We have already shown that $1 - w_k(x) \leq c_3 u_k(x)$ for some constant $c_3 > 0$. Furthermore,

$$u_k(x) = P^x(Z_k > 0) \leq E^x[Z_k] = b^k P^x(\tau_0 > k) .$$

Since $P^x(\tau_0 > k) \leq E\left[e^{\nu(x+S_k)}\right] = e^{\nu x}\phi(\nu)^k$, we have

$$1 - w_k(x) \leq c_3 e^{\nu x} \phi(\nu)^k . \tag{3.4}$$

This implies that

$$\left\{\prod_{k=K}^{n} w_k(S_{n-k}) < 1-\varepsilon,\, S_{n-k} \leq M + k^{2/3},\, K \leq k \leq n\right\} = \emptyset$$

for K greater than some K_1. It remains to bound the probability

$$P^x\left(\exists\, K \leq k < n \text{ such that } S_n - S_{n-k} \leq -k^{2/3},\, S_n \leq M \,\middle|\, T_0 > n\right) .$$

We use the Kolmogorov's extension theorem to define the probability $\mathbb{Q}$ such that for any n,

$$d\mathbb{Q}|_{\mathcal{F}_n} \quad := \quad \frac{e^{\nu(S_n - S_0)}}{\phi(\nu)^n} dP|_{\mathcal{F}_n} ,$$

where $\mathcal{F}_n$ is the sigma-algebra generated by $S_0, S_1, \ldots, S_n$. Under the probability $\mathbb{Q}$, the random walk S is centered. We can write

$$\begin{aligned}
& P^x\left(\exists\, K \le k < n \text{ such that } S_n - S_{n-k} \le -k^{2/3},\, S_n \le M \,\middle|\, \tau_0 > n\right) \\
= \;& E^x_{\mathbb{Q}}\left[e^{-\nu S_n}, \exists\, K \le k < n \text{ such that } S_n - S_{n-k} \le -k^{2/3},\, S_n \le M,\, \tau_0 > n\right] \frac{e^{\nu x}\phi(\nu)^n}{P^x\left(\tau_0 > n\right)} \\
\le \;& \mathbb{Q}^x\left(\exists\, K \le k < n \text{ such that } S_n - S_{n-k} \le -k^{2/3},\, S_n \le M,\, \tau_0 > n\right) \frac{e^{\nu x}\phi(\nu)^n}{P^x\left(\tau_0 > n\right)}\,.
\end{aligned}$$

We know by equation (1.4) that

$$P^x\left(\tau_0 > n\right) \sim C_2 e^{\nu x}\widetilde{V}(x)\frac{\phi(\nu)^n}{n^{3/2}}\,.$$

Therefore, it is sufficient to show that there exists K_2 such that for $K \ge K_2$,

$$\mathbb{Q}^x\left(\exists\, K \le k < n \text{ such that } S_n - S_{n-k} \le -k^{2/3},\, S_n \le M,\, \tau_0 > n\right) \le \frac{\varepsilon/2}{n^{3/2}}\,, \tag{3.5}$$

for n large enough. First, we propose to bound the probability $P(R_k \ge \ell^{2/3})$ where $R_k = Y_1 + \ldots + Y_k$ is any random walk of mean zero admitting exponential moments, and where $\ell \ge k \ge 0$. To this end, we follow the proof of Lemma 6.3 in [DGPZ02]. Let $c_4 > 0$, $c_5 > 0$ be such that for any $z \ge 0$,

$$P(|Y_1| > z) \le c_4 e^{-c_5 z}\,.$$

We observe that

$$\begin{aligned}
P\left(R_k \ge \ell^{2/3}\right) &\le kP\left(Y_1 \ge \sqrt{\ell}\right) + P\left(R_k \ge \ell^{2/3},\, Y_i < \sqrt{\ell},\, i = 1\ldots k\right) \\
&\le c_4 k e^{-c_5\sqrt{\ell}} + e^{-\ell^{1/6}} E\left[e^{R_k/\sqrt{\ell}}\mathbb{1}_{\{Y_i < \sqrt{\ell},\, 1 \le i \le k\}}\right] \\
&\le c_4 k e^{-c_5\sqrt{\ell}} + e^{-\ell^{1/6}} E\left[e^{Y_1/\sqrt{\ell}}\mathbb{1}_{\{Y_1 < \sqrt{\ell}\}}\right]^k.
\end{aligned}$$

The inequality $e^u \le 1 + u + u^2$ for $u \le 1$ implies that

$$\begin{aligned}
E\left[e^{Y_1/\sqrt{\ell}}\mathbb{1}_{\{Y_1 < \sqrt{\ell}\}}\right] &\le 1 + E\left[\frac{Y_1}{\sqrt{\ell}}\mathbb{1}_{\{Y_1 < \sqrt{\ell}\}}\right] + E\left[\frac{Y_1^2}{\ell}\mathbb{1}_{\{Y_1 < \sqrt{\ell}\}}\right] \\
&\le 1 + \frac{E\left[Y_1^2\right]}{\ell}\,.
\end{aligned}$$

Since $\ell \ge k$, we get

$$\begin{aligned}
E\left[e^{Y_1/\sqrt{\ell}}\mathbb{1}_{\{Y_1 < \sqrt{\ell}\}}\right]^k &\le \left(1 + \frac{E\left[Y_1^2\right]}{\ell}\right)^k \\
&\le \left(1 + \frac{E\left[Y_1^2\right]}{\ell}\right)^\ell \\
&\le \exp\left(E\left[Y_1^2\right]\right)\,.
\end{aligned}$$

Therefore, applying this to the probability $\mathbb{Q}$ and the random walk $-S$, we can find constants $c_6 > 0$, $c_7 > 0$ such that for $\ell \geq k$,

$$\mathbb{Q}^0 \left(S_k \leq -\ell^{\,2/3} \right) \leq c_6 \exp(-c_7 \ell^{\,1/6}) \,, \tag{3.6}$$

To prove equation (3.5), we will show that there exists K_3 such that, for $K \geq K_3$,

$$\begin{aligned}
(3.7) \quad & \mathbb{Q}^x \left(\exists\, k < n^{1/2} \text{ such that } S_n - S_{n-k} \leq -(n-k)^{1/3},\, S_n \leq M, \tau_0 > n \right) \\
= \quad & o(n^{3/2}) \,, \\
(3.8) \quad & \mathbb{Q}^x \left(\exists\, k \in [n^{1/2}, n] \text{ such that } S_n - S_{n-k} \leq -k^{2/3},\, S_n \leq M, \tau_0 > n \right) \\
= \quad & o(n^{3/2}) \,, \\
(3.9) \quad & \mathbb{Q}^x \left(\exists\, K \leq k < n^{1/2} \text{ such that } S_n - S_{n-k} \in [-(n-k)^{1/3}, -k^{2/3}],\, S_n \leq M, \tau_0 > n \right) \\
\leq \quad & \frac{\varepsilon/4}{n^{3/2}} \,,
\end{aligned}$$

where we have set that $[i,j] = \emptyset$ if $j < i$.

We first prove (3.7) and (3.8). Observe that

$$\begin{aligned}
& \mathbb{Q}^x \left(\exists\, k < n^{1/2} \text{ such that } S_n - S_{n-k} \leq -(n-k)^{1/3},\, S_n \leq M,\, \tau_0 > n \right) \\
\leq \; & \mathbb{Q}^x \left(\exists\, k < n^{1/2} \text{ such that } S_n - S_{n-k} \leq -(n-n^{1/2})^{1/3} \right) \\
\leq \; & \sum_{k<n^{1/2}} \mathbb{Q}^0 \left(S_k \leq -(n-n^{1/2})^{1/3} \right)
\end{aligned}$$

Use (3.6) to get equation (3.7). Similarly,

$$\begin{aligned}
& \mathbb{Q}^x \left(\exists\, n^{1/2} \leq k < n \text{ such that } S_n - S_{n-k} \leq -k^{2/3},\, S_n \leq M,\, \tau_0 > n \right) \\
\leq \; & \sum_{k=n^{1/2}}^{n} \mathbb{Q}^0 \left(S_k \leq -k^{2/3} \right) \,.
\end{aligned}$$

Applying again (3.6) gives (3.8).

To prove equation (3.9), let $k < n$ and define $a(k) := k^{2/3}$, $b(k) := M + (1+k)^{2/3}$, and for any $\ell > k$,

$$\begin{aligned}
a(\ell) \; &:= \; M + \ell^{\,2/3} \,, \\
b(\ell) \; &:= \; M + (1+\ell)^{\,2/3} \,.
\end{aligned}$$

We notice that

$$\begin{aligned}
&\mathbb{Q}^x(S_n - S_{n-k} \in [-(n-k)^{1/3}, -k^{2/3}],\, S_n \le M,\, \tau_0 > n)\\
\le\ & \sum_{\ell=k}^{(n-k)^{1/2}} \mathbb{Q}^x\left(S_n - S_{n-k} \le -\ell^{2/3},\, S_{n-k} \in [a(\ell), b(\ell)],\, \tau_0 > n\right)\\
\le\ & \sum_{\ell=k}^{(n-k)^{1/2}} \mathbb{Q}^x\left(S_n - S_{n-k} \le -\ell^{2/3},\, S_{n-k} \in [a(\ell), b(\ell)],\, \tau_0 > n-k\right)\,,
\end{aligned}$$

where $\sum_i^j := 0$ if $j < i$. By the Markov property at time $n-k$, we deduce that

$$\begin{aligned}
&\mathbb{Q}^x(S_n - S_{n-k} \in [-(n-k)^{1/3}, -k^{2/3}],\, S_n \le M,\, \tau_0 > n)\\
\le\ & \sum_{\ell=k}^{(n-k)^{1/2}} \mathbb{Q}^x\left(S_{n-k} \in [a(\ell), b(\ell)],\, \tau_0 > n-k\right)\mathbb{Q}^0\left(S_k \le -\ell^{2/3}\right)\\
\le\ & \sum_{\ell=k}^{(n-k)^{1/2}} \mathbb{Q}^x\left(S_{n-k} \in [a(\ell), b(\ell)],\, \tau_0 > n-k\right) c_6 \exp(-c_7 \ell^{1/6})\,, \qquad (3.10)
\end{aligned}$$

by equation (3.6). We would like to estimate $\mathbb{Q}\left(S_m \in [a(\ell), b(\ell)],\, \tau_0 > m\right)$. Recall the definition of $\widetilde{V}$ in (1.3). We observe that $\widetilde{V}(u)$ can be seen as the expected hitting time of the level u by a random walk with nonnegative increments. Therefore, there exist constants $c_8 > 0$, $c_9 > 0$ such that

$$\widetilde{V}(u) \le c_8 + c_9 u\,.$$

By Theorem 3 of Vatutin and Wachtel [VW08], we have for any $\delta > 0$, and any $v \in [0, m^{1/3} + M]$,

$$\mathbb{Q}^0\left(S_m \in [v, v+\delta],\, \tau_0 > m\right) = \frac{1}{\sigma\sqrt{2\pi}} \frac{\int_v^{v+\delta} \widetilde{V}(u)du}{m^{3/2}} (1 + f_\delta(m, v)) \qquad (3.11)$$

where $f_\delta(m,v) \le g_\delta(m)$ for some function g_δ which tends to 0 at infinity. Take $\delta := 1 + M$. Equation (3.11) gives that there exists an integer i such that

$$\mathbb{Q}^0\left(S_i \in [x, x+\delta],\, \tau_0 > i\right) > 0.$$

We observe that

$$\mathbb{Q}^0\left(S_{m+i} \in [v, v+2\delta],\, \tau_0 > m+i\right) \ge \mathbb{Q}^0\left(S_i \in [x, x+\delta],\, \tau_0 > i\right)\mathbb{Q}^x\left(S_m \in [v, v+\delta],\, \tau_0 > m\right)$$

and, using again (3.11), we obtain that there exist constants c_{13} and c_{14} such that for any integer m and any $v \in [0, m^{1/3} + M]$

$$\mathbb{Q}^x \left(S_m \in [v, v+\delta] \,,\, \tau_0 > m \right) \leq \frac{c_{13} + c_{14} v}{m^{3/2}} \,.$$

It yields that, (beware that $b(\ell) \leq a(\ell) + \delta$),

$$\begin{aligned}
&\sum_{\ell=k}^{(n-k)^{1/2}} \mathbb{Q}^x \left(S_{n-k} \in [a(\ell), b(\ell)] \,,\, \tau_0 > n-k \right) \exp(-c_7 \ell^{1/6}) \\
\leq\;& \frac{1}{(n-k)^{3/2}} \sum_{\ell=k}^{(n-k)^{1/2}} \exp(-c_7 \ell^{1/6}) \left(c_{13} + c_{14} \ell^{2/3} \right) \\
\leq\;& c_{15} \frac{\exp(-c_{17}\, k^{1/6})}{(n-k)^{3/2}} \,,
\end{aligned}$$

with for example $c_{17} = c_7/2$. Therefore, equation (3.10) becomes

$$\mathbb{Q}^x(S_n - S_{n-k} \in [-(n-k)^{1/3}, -k^{2/3}],\, S_n \leq M,\, \tau_0 > n) \leq c_{16} \frac{\exp(-c_{17}\, k^{1/6})}{(n-k)^{3/2}} \,.$$

Since $n \geq k+1$, we have that $\left(1 - \frac{k}{n}\right)^{3/2} \geq \left(\frac{1}{1+k}\right)^{3/2}$, which gives

$$\mathbb{Q}^x(S_n - S_{n-k} \in [-(n-k)^{1/3}, -k^{2/3}],\, S_n \leq M,\, \tau_0 > n) \leq \frac{r_k}{n^{3/2}} \,,$$

with

$$r_k := c_{16} \exp(-c_{17}\, k^{1/6})\, (1+k)^{3/2} \,.$$

It yields that

$$\begin{aligned}
&\mathbb{Q}^x \left(\exists\, K \leq k < n^{1/2} \text{ such that } S_n - S_{n-k} \in [-(n-k)^{1/3}, -k^{2/3}],\, S_n \leq M, \tau_0 > n \right) \\
\leq\;& \frac{\sum_{k \geq K} r_k}{n^{3/2}} \,.
\end{aligned}$$

We obtain equation (3.9) by choosing K large enough. This completes the proof. □

Corollary 3.6. *We have*

$$\lim_{n \to \infty} E^x \left[\prod_{k=0}^{n} (1 - w_k(S_{n-k})) \,\middle|\, \tau_0 > n \right] = \inf_{K \geq 1} a_K$$

and $\inf_{K \geq 1} a_K > 0$.

Proof. We recall that

$$\lim_{n\to\infty} E^x\left[\prod_{k=0}^{K} w_k(S_{n-k})\,\Big|\,\tau_0 > n\right] = a_K\,.$$

By Lemma 3.5, we deduce that for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} E^x\left[\prod_{k=0}^{n} w_k(S_{n-k})\,\Big|\,\tau_0 > n\right] \geq (1-\varepsilon)\inf_{K\geq 1} a_K - \varepsilon\,.$$

Let ε tend to 0 to get

$$\liminf_{n\to\infty} E^x\left[\prod_{k=0}^{n} w_k(S_{n-k})\,\Big|\,\tau_0 > n\right] \geq \inf_{K\geq 1} a_K\,.$$

Combine it with Corollary 3.4 to get the first equation. Let $\mu > 0$. We write as before (see the beginning of the proof of Lemma 3.5)

$$P^x\left(\prod_{k=0}^{n} w_k(S_{n-k}) < \mu\,\Big|\,\tau_0 > n\right) \leq P^x(A_1\,|\,\tau_0 > n) + P^x(A_2\,|\,\tau_0 > n) + P^x(A_3\,|\,\tau_0 > n)$$

with

$$\begin{aligned}
A_1 &:= \{S_n \geq M\}\\
A_2 &:= \left\{\exists\, K \leq k < n \text{ such that } S_n - S_{n-k} \leq -k^{2/3},\, S_n \leq M\right\}\\
A_3 &:= \left\{\prod_{k=0}^{n} w_k(S_{n-k}) < \mu,\, S_{n-k} \leq M + k^{2/3},\, K \leq k \leq n\right\}.
\end{aligned}$$

Let $\eta > 0$. We showed that there exist $M := M(\eta)$ and $K := K(\eta)$ such that $P^x(A_1\,|\,\tau_0 > n) + P^x(A_2\,|\,\tau_0 > n) \leq \eta$. On the event A_3, we have by (3.4)

$$\prod_{k=K(\eta)+1}^{n} w_k(S_{n-k}) \geq \prod_{k=K(\eta)+1}^{\infty}\left(1 - c_3 e^{\nu(M+k^{2/3})}\phi(\nu)^k\right) =: F(\eta)\,.$$

We take care of choosing $K(\eta)$ big enough to have $F(\eta) > 0$. Besides, we emphasize that η, M, K and $F(\eta)$ do not depend on the value of μ. We have

$$P^x(A_3\,|\,\tau_0 > n) \leq P^x\left(\prod_{k=0}^{K(\eta)} w_k(S_{n-k}) < \mu/F(\eta)\,\Big|\,\tau_0 > n\right)\,.$$

We observe that

$$P^x\left(\prod_{k=0}^{K(\eta)} w_k(S_{n-k}) < \mu/F(\eta),\, \tau_0 > n\right) \quad =: \quad E^x\left[g_{\eta,\mu}(S_{n-K(\eta)}),\, \tau_0 > n - K(\eta)\right]$$

by the Markov property, where $g_{\eta,\mu}(z) := P^z\left(\prod_{k=0}^{K(\eta)} w_k(S_{K(\eta)-k}) < \mu/F(\eta),\, \tau_0 > K(\eta)\right)$. We use Lemma 3.2 to see that $P^x\left(\prod_{k=0}^{K(\eta)} w_k(S_{n-k}) < \mu/F(\eta) \,\middle|\, \tau_0 > n\right)$ tends to $\gamma^{-K(\eta)}E[g_{\eta,\mu}(S^*)]$. Let $\mu > 0$ small enough to have $\gamma^{-K(\eta)}E[g_{\eta,\mu}(S^*)] \leq \eta$. We have

$$\begin{aligned} E^x\left[\prod_{k=0}^{n} w_k(S_{n-k}) \,\middle|\, \tau_0 > n\right] &\geq \mu P^x\left(\prod_{k=0}^{n} w_k(S_{n-k}) > \mu \,\middle|\, \tau_0 > n\right) \\ &\geq \mu\left(1 - \eta - P^x\left(\prod_{k=0}^{K(\eta)} w_k(S_{n-k}) < \mu/F(\eta) \,\middle|\, \tau_0 > n\right)\right). \end{aligned}$$

We take the limit to obtain

$$\inf_{K\geq 1} a_K \geq \mu(1-2\eta) > 0$$

if η is taken strictly smaller than 1/2. $\square$

Proposition 3.1 follows from Corollary 3.6.

4 The critical case

Before the proof of Theorem 1.2, we mention a small deviations result of Mogul'skii [Mog75], which gives the probability for a random walk to stay between two curves. Let $\mathcal{C}[0,1]$ denote the set of continuous functions defined on $[0,1]$.

Theorem(Mogul'skii) *: Let $\xi_1,\, \xi_2,\, \ldots$ be i.i.d variables with $E[\xi_1] = 0$, $E[\xi_1^2] = \sigma^2 \in (0,\infty)$. Let $(x_n,\, n \geq 0)$ be a sequence of positive numbers such that*

$$\begin{aligned} \lim_{n\to\infty} x_n &= +\infty \\ \lim_{n\to\infty} \frac{x_n}{\sqrt{n}} &= 0\,. \end{aligned}$$

Define for any $t \in [0,1]$

$$s_n(t) := \frac{\xi_1 + \xi_2 + \ldots + \xi_k}{x_n} \qquad \text{for} \qquad k/n \leq t < (k+1)/n\,.$$

Then, for any $L_1,\ L_2 \in \mathcal{C}[0,1]$, with $L_2(t) > L_1(t)$, we have

$$\ln\left(P(L_2(t) > s_n(t) > L_1(t),\ \ \forall\, 0 \le t \le 1)\right) \sim -\frac{\sigma^2\pi^2}{2} n x_n^{-2} \int_0^1 \frac{dt}{\left(L_2(t) - L_1(t)\right)^2}\,. \tag{4.1}$$

We will take $x_n = n^{1/3}$, $L_1(t) = -r_1$ and $L_2(t) = d_1 \times (1 - d_2 t)^{1/3} - r_2$ for some $0 \le r_2 \le r_1$, $d_1 > 0$ and $0 < d_2 < 1$. Recall that $\mathbb{Q}$, defined by

$$d\mathbb{Q}|_{\mathcal{F}_n} \;\; := \;\; \frac{e^{\nu(S_n - S_0)}}{\phi(\nu)^n} d\mathbb{P}|_{\mathcal{F}_n}\,,$$

is a probability under which S_n is centered. Let us define $\sigma^2 := E_{\mathbb{Q}}[X_1^2]$. By (4.1), it gives that for any $z > 0$

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^{1/3}} \ln\left(\mathbb{Q}^z\left(L\left(k/n\right) > \frac{S_k}{n^{1/3}} > -r_1\ ,\ \forall\, 0 \le k \le n\right)\right) \\ &= -\frac{3\sigma^2\pi^2}{2d_1^2} \times H(r_1 - r_2, d_1, d_2) \end{aligned} \tag{4.2}$$

with

$$L(t) := d_1 \times (1 - d_2 t)^{1/3} - r_2\,.$$

and $H(u,v,w)$ a continuous function on $[0,+\infty[\times]0,+\infty[\times[0,1]$, such that $H(0,v,1) = 1$ for any $v > 0$.

We turn to the proof of Theorem 1.2, which borrows some idea from Kesten [Kes78] who treated the case of branching Brownian motion. We assume that $(\gamma =)E[e^{\nu X}] = \frac{1}{b}$ and let $x > 0$. Let $(f_k,\ k \ge 0)$ be a sequence of positive reals. The value of f_k will be made precise later on. For any $|u| = k$, we say that $u \in \mathcal{S}$ if for any generation $\ell \le k$, the ancestor u_ℓ of u verifies $0 < S(u_\ell) < f_\ell$. We introduce

$$Z_n(\mathcal{S}) := \sum_{|y|=n} \mathbb{1}_{\{y \in \mathcal{S}\}}\,.$$

In words, we are interested by the number of particles that have always been below the curve f. For the underlying one-dimensional random walk $(S_k, k \ge 0)$, we then define

$$\tau_f := \inf\{k \ge 1\ :\ S_k \ge f_k \text{ or } S_k \le 0\}\,. \tag{4.3}$$

Proposition 4.1. *The following two inequalities hold :*

$$P^x(Z_n > 0) \;\geq\; \frac{e^{\nu x}\mathbb{Q}^x(\tau_f > n)e^{-2\nu f_n}}{1 + \sum_{k=0}^{n-1} \sup_{0\leq z\leq f_k} \{e^{\nu z}\mathbb{Q}^z\left(\tau_f > n-k\right)\}}, \tag{4.4}$$

$$P^x(Z_n > 0) \;\leq\; e^{\nu x}\left(\mathbb{Q}^x(\tau_f > n) + \sum_{k=1}^{n} \mathbb{Q}^x(\tau_0 > k)e^{-\nu f_k}\right). \tag{4.5}$$

Proof. By the Cauchy-Schwarz inequality,

$$E^x\left[Z_n(\mathcal{S})\right]^2 \leq E^x\left[Z_n(\mathcal{S})^2\right] P^x\left(Z_n(\mathcal{S}) > 0\right),$$

which yields

$$P^x(Z_n > 0) \geq P^x(Z_n(\mathcal{S}) > 0) \geq \frac{E^x[Z_n(\mathcal{S})]^2}{E^x[Z_n(\mathcal{S})^2]}. \tag{4.6}$$

We observe that

$$Z_n(\mathcal{S})^2 \;=\; \sum_{|u|=n} \mathbb{I}_{\{u\in\mathcal{S}\}} Z_n(\mathcal{S}). \tag{4.7}$$

For any $v \in \mathcal{S}$, we define

$$Z_n^v(\mathcal{S}) := \sum_{|u|=n, u>v} \mathbb{I}_{\{u\in\mathcal{S}\}}.$$

Moreover, if w is a child of v, and v_i denotes the i-th child of v, we set

$$Z_n^v(\mathcal{S}, w) := \sum_{i, v_i\neq w} Z_n^{v_i}(\mathcal{S})$$

which stands for the number of descendants of v in generation n who have never been beyond the curve f neither below zero and who are not descendant of w. Let $|u| = n$, and let u_ℓ be as previously the ancestor at generation ℓ. We have

$$Z_n(\mathcal{S}) \;=\; 1 + \sum_{k=0}^{n-1} Z_n^{u_k}(\mathcal{S}, u_{k+1}),$$

from which equation (4.7) becomes

$$\begin{aligned} Z_n(\mathcal{S})^2 &= Z_n(\mathcal{S}) + \sum_{|u|=n}\sum_{k=0}^{n-1} \mathbb{I}_{\{u\in\mathcal{S}\}} Z_n^{u_k}(\mathcal{S}, u_{k+1}) \\ &= Z_n(\mathcal{S}) + \sum_{k=1}^{n}\sum_{|v|=k} Z_n^v(\mathcal{S}) Z_n^{\overleftarrow{v}}(\mathcal{S}, v), \end{aligned}$$

where $\overleftarrow{v}$ stands for the father of v. Conditionally on $\overleftarrow{v} \in \mathcal{S}$ and $S(\overleftarrow{v})$, the random variables $Z_n^v(\mathcal{S})$ and $Z_n^{\overleftarrow{v}}(\mathcal{S}, v)$ are independent. This implies that

$$\begin{aligned}
& E^x[Z_n(\mathcal{S})^2] \\
\leq \;& E^x[Z_n(\mathcal{S})] + \sum_{k=1}^{n} \sum_{|v|=k} E^x \left[E^x[Z_n^v(\mathcal{S}) \,|\, \overleftarrow{v} \in \mathcal{S}, S(\overleftarrow{v})] E^x[Z_n^{\overleftarrow{v}}(\mathcal{S}, v) \,|\, \overleftarrow{v} \in \mathcal{S}, S(\overleftarrow{v})] \right] \\
\leq \;& E^x[Z_n(\mathcal{S})] + \sum_{k=1}^{n} \sum_{|v|=k} E^x[E^x[Z_n^{\overleftarrow{v}}(\mathcal{S}) \,|\, \overleftarrow{v} \in \mathcal{S}, S(\overleftarrow{v})]^2] \\
= \;& E^x[Z_n(\mathcal{S})] + \sum_{k=0}^{n-1} \sum_{|v|=k} E^x[E^x[Z_n^v(\mathcal{S}) \,|\, v \in \mathcal{S}, S(v)]^2] \,. \qquad (4.8)
\end{aligned}$$

For $|v| = k$, we notice that

$$E^x \left[Z_n^v(\mathcal{S}) \,|\, v \in \mathcal{S},\, S(v) \right] = \mathbb{1}_{\{v \in \mathcal{S}\}} b^{n-k} P^{S(v)} \left(\tau_f > n-k \right) \,, \qquad (4.9)$$

where τ_f is defined in (4.3). We have for any $a > 0$ and $\ell \geq 0$,

$$\begin{aligned}
P^a \left(\tau_f > \ell \right) &= e^{\nu a} E[e^{\nu X_1}]^\ell E_{\mathbb{Q}}^a \left[\mathbb{1}_{\{\tau_f > \ell\}} e^{-\nu S_\ell} \right] \\
&\leq e^{\nu a} E[e^{\nu X_1}]^\ell \mathbb{Q}^a \left(\tau_f > \ell \right) \\
&= e^{\nu a} b^{-\ell} \mathbb{Q}^a \left(\tau_f > \ell \right) \,.
\end{aligned}$$

Therefore, equation (4.9) says that

$$E^x \left[Z_n^v(\mathcal{S}) \,|\, v \in \mathcal{S},\, S(v) \right] \leq \mathbb{1}_{\{v \in \mathcal{S}\}} e^{\nu S(v)} \mathbb{Q}^{S(v)} \left(\tau_f > n-k \right) \,. \qquad (4.10)$$

From (4.8), we deduce that

$$E^x[Z_n^2(\mathcal{S})] \;\leq\; E^x \left[Z_n(\mathcal{S}) \right] + \sum_{k=0}^{n-1} \sum_{|v|=k} E^x \left[\mathbb{1}_{\{v \in \mathcal{S}\}} \left(e^{\nu S(v)} Q^{S(v)} \left(\tau_f > n-k \right) \right)^2 \right] \,.$$

We notice that

$$E^x[Z_n(\mathcal{S})] = b^n P^x(\tau_f > n) = e^{\nu x} E_{\mathbb{Q}}^x \left[e^{-\nu S_n} \mathbb{1}_{\{\tau_f > n\}} \right] \leq e^{\nu x} \mathbb{Q}^x(\tau_f > n) \,. \qquad (4.11)$$

Consequently,

$$\begin{aligned}
E^x[Z_n^2(\mathcal{S})] &\leq e^{\nu x} \mathbb{Q}^x(\tau_f > n) + \sum_{k=0}^{n-1} b^k E^x \left[\mathbb{1}_{\{\tau_f > k\}} \left(e^{\nu S_k} \mathbb{Q}^{S_k} \left(\tau > n-k \right) \right)^2 \right] \\
&= e^{\nu x} \mathbb{Q}^x(\tau_f > n) + e^{\nu x} \sum_{k=0}^{n-1} E_{\mathbb{Q}}^x \left[\mathbb{1}_{\{\tau_f > k\}} e^{\nu S_k} \left(\mathbb{Q}^{S_k} \left(\tau_f > n-k \right) \right)^2 \right] \,.
\end{aligned}$$

For any k, we have

$$\begin{aligned} & E_{\mathbb{Q}}^x\left[\mathbb{1}_{\{\tau_f>k\}}e^{\nu S_k}\left(\mathbb{Q}^{S_k}\left(\tau_f>n-k\right)\right)^2\right] \\ \leq\ & E_{\mathbb{Q}}^x\left[\mathbb{1}_{\{\tau_f>k\}}\mathbb{Q}^{S_k}\left(\tau_f>n-k\right)\right]\sup_{0\leq z\leq f_k}\left\{e^{\nu z}\mathbb{Q}^z\left(\tau_f>n-k\right)\right\}. \end{aligned}$$

Since $E_{\mathbb{Q}}^x\left[\mathbb{1}_{\{\tau_f>k\}}\mathbb{Q}^{S_k}\left(\tau_f>n-k\right)\right]=\mathbb{Q}^x(\tau_f>n)$, this gives

$$E_{\mathbb{Q}}^x\left[\mathbb{1}_{\{\tau_f>k\}}e^{\nu S_k}\left(\mathbb{Q}^{S_k}\left(\tau_f>n-k\right)\right)^2\right]\leq\mathbb{Q}^x(\tau_f>n)\sup_{0\leq z\leq f_k}\left\{e^{\nu z}\mathbb{Q}^z\left(\tau_f>n-k\right)\right\}.$$

Hence,

$$E^x[Z_n(\mathcal{S})^2]\leq e^{\nu x}\mathbb{Q}^x(\tau_f>n)\left(1+\sum_{k=0}^{n-1}\sup_{0\leq z\leq f_k}\left\{e^{\nu z}\mathbb{Q}^z\left(\tau_f>n-k\right)\right\}\right).$$

Then, by (4.6) and (4.11),

$$\begin{aligned} P^x(Z_n>0) &\geq \frac{e^{\nu x}E_{\mathbb{Q}}^x\left[e^{-\nu S_n}\mathbb{1}_{\{\tau_f>n\}}\right]^2}{\mathbb{Q}^x(\tau_f>n)\left(1+\sum_{k=0}^{n-1}\sup_{0\leq z\leq f_k}\left\{e^{\nu z}\mathbb{Q}^z\left(\tau_f>n-k\right)\right\}\right)} \\ &\geq \frac{e^{-2\nu f_n}e^{\nu x}\mathbb{Q}^x(\tau_f>n)}{1+\sum_{k=0}^{n-1}\sup_{0\leq z\leq f_k}\left\{e^{\nu z}\mathbb{Q}^z\left(\tau_f>n-k\right)\right\}}. \end{aligned}$$

This is equation (4.4).

Turning to the proof of (4.5), we observe that

$$\{Z_n>0\}\quad\subset\quad\{Z_n(\mathcal{S})>0\}\cup\bigcup_{k=1}^{n}E_k.$$

where E_k is the event that a particle u surviving at time n went beyond the curve f for the first time at time $k<n$. We say then that u is k-good. We already have

$$P^x(Z_n(\mathcal{S})>0)\leq E^x[Z_n(\mathcal{S})]\leq e^{\nu x}\mathbb{Q}^x(\tau_f>n).$$

For any $k\leq n$, we observe that

$$P^x\left(E_k\right)\leq E^x\left[\sum_{|u|=k}\mathbb{1}_{\{u\text{ is }k-\text{good}\}}\right]=b^kP^x\left(\tau_0>n,\,\tau_f=k\right)\leq b^kP^x\left(\tau_0>k,\,\tau_f=k\right).$$

This leads to

$$P^x\left(E_k\right)\leq e^{\nu x}E_{\mathbb{Q}}^x\left[\mathbb{1}_{\{\tau_0>k\}}\mathbb{1}_{\{\tau_f=k\}}e^{-\nu S_k}\right]\leq e^{\nu x}\mathbb{Q}^x(\tau_0>k)e^{-\nu f_k}.$$

Finally,

$$P^x(Z_n > 0) \leq e^{\nu x} \left(\mathbb{Q}^x(\tau_f > n) + \sum_{k=1}^{n} \mathbb{Q}^x(\tau_0 > k) e^{-\nu f_k} \right) .$$

It completes the proof of (4.5). □

Let $d_1 > 0$, $1 > d_2 > 0$, and define

$$f_k := d_1 \left(n - d_2 k\right)^{1/3} .$$

Observe that $0 \leq f_k \leq d_1 n$. Let L and M be two (large) integers. We introduce

$$\begin{aligned} k_i &:= \lfloor in/L \rfloor , \quad i \in [\![0, L]\!] , \\ x_j &:= \lfloor d_1 nj/M \rfloor , \quad j \in [\![0, M]\!] . \end{aligned}$$

Let us proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. We first prove the upper bound. By Proposition 4.1, it suffices to bound $R(n) := \mathbb{Q}^x(\tau_f > n) + \sum_{k=1}^{n} \mathbb{Q}^x(\tau_f > k) e^{-\nu f_k}$. We observe that

$$R(n) \quad \leq \quad \mathbb{Q}^x(\tau_f > n) + (1 + \frac{n}{L}) \sum_{i=0}^{L-1} \mathbb{Q}^x(\tau_f > k_i) e^{-\nu f_{k_{i+1}}} .$$

Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n^{1/3}} \ln\left(P^x(Z_n > 0)\right) &\leq \limsup_{n\to\infty} \frac{1}{n^{1/3}} \ln R(n) \\ &\leq \max(R_1, R_2) =: R_1 \vee R_2 , \end{aligned}$$

with

$$\begin{aligned} R_1 &:= \limsup_{n\to\infty} \frac{1}{n^{1/3}} \ln \mathbb{Q}^x(\tau_f > n) , \\ R_2 &:= \max_{0 \leq i \leq L-1} \limsup_{n\to\infty} \frac{1}{n^{1/3}} \ln \left(\mathbb{Q}^x(\tau_f > k_i) e^{-\nu f_{k_{i+1}}} \right) . \end{aligned}$$

By equation (4.2),

$$R_1 = -\frac{3\sigma^2\pi^2}{2d_1^2} \times H(0, d_1, d_2) .$$

Similarly, for any $i \in [\![0, L-1]\!]$,

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \ln(\mathbb{Q}^x(\tau_f > k_i)e^{-\nu f_{k_{i+1}}}) = -\frac{3\sigma^2\pi^2}{2d_1^2}\left(\frac{i}{L}\right)^{1/3} \times H(0,d_1,d_2) - \nu d_1 \left(1 - d_2\frac{i+1}{L}\right)^{1/3}.$$

Therefore, we get

$$\begin{aligned} R_2 &= \max_{0\le i\le L-1}\left\{-\frac{3\sigma^2\pi^2}{2d_1^2}\left(\frac{i}{L}\right)^{1/3} H(0,d_1,d_2) - \nu d_1\left(1-d_2\frac{i+1}{L}\right)^{1/3}\right\} \\ &\le \sup_{t\in[0,1]}\left\{-\frac{3\sigma^2\pi^2}{2d_1^2}t^{1/3}H(0,d_1,d_2) - \nu d_1\left(1-\frac{d_2}{L}-d_2 t\right)^{1/3}\right\} =: S(d_1,d_2,1/L)\,. \end{aligned}$$

Finally,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} P^x(Z_n>0) \le \left(-\frac{3\sigma^2\pi^2}{2d_1^2}H(0,d_1,d_2)\right) \vee S(d_1,d_2,1/L)\,.$$

Let L and d_2 respectively tend to ∞ and 1 to obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n^{1/3}} P^x(Z_n>0) &\le \left(-\frac{3\sigma^2\pi^2}{2d_1^2}\right) \vee \left(\sup_{t\in[0,1]}\left\{-\frac{3\sigma^2\pi^2}{2d_1^2}t^{1/3} - \nu d_1\,(1-t)^{1/3}\right\}\right) \\ &= \sup_{t\in[0,1]}\left\{-\frac{3\sigma^2\pi^2}{2d_1^2}t^{1/3} - \nu d_1\,(1-t)^{1/3}\right\}. \end{aligned}$$

We take $d_1 = \left(\frac{3\sigma^2\pi^2}{2\nu}\right)^{1/3}$ to obtain

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} P^x(Z_n>0) \le -\left(\frac{3\sigma^2\pi^2\nu^2}{2}\right)^{1/3},$$

which completes the proof of the upper bound in Theorem 1.2. We prove the lower bound. By Proposition 4.1,

$$\begin{aligned} &\ln(P^x(Z_n>0)) - \nu x \\ \ge\ & \ln(\mathbb{Q}^x(\tau_f>n)) - \ln\left(1+\sum_{k=0}^{n-1}\sup_{0\le z\le f_k}\{e^{\nu z}\mathbb{Q}^z(\tau_f>n-k)\}\right) - 2\nu d_1(1-d_2)n^{1/3} \\ \ge\ & \ln(\mathbb{Q}^x(\tau_f>n)) - \ln\left(1+(1+\frac{n}{L})\sum_{i=0}^{L-1}e^{\nu f_{k_{i+1}}}\sup_{0\le z\le f_{k_{i+1}}}\mathbb{Q}^z(\tau_f>n-k_i)\right) \\ & -2\nu d_1(1-d_2)n^{1/3}\,. \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \ln(P^x(Z_n > 0)) \geq T_1 - T_2 - 2\nu d_1(1-d_2)$$

with

$$\begin{aligned} T_1 &:= \liminf_{n\to\infty} \frac{1}{n^{1/3}} \ln \mathbb{Q}^x(\tau_f > n) = -\frac{3\sigma^2\pi^2}{2d_1^2} H(0,d_1,d_2)\,, \\ T_2 &:= \max_{0\leq i\leq L-1} \left\{ d_1\nu\left(1-\frac{i+1}{L}\right)^{1/3} + \limsup_{n\to\infty} \frac{1}{n^{1/3}} \ln\left(\sup_{0\leq z\leq f_{k_{i+1}}} \mathbb{Q}^z\left(\tau_f > n-k_i\right)\right)\right\}. \end{aligned}$$

Furthermore, we observe that for any reals w, y and any integer ℓ,

$$\sup_{w\leq z\leq y} \{\mathbb{Q}^z\left(\tau_f > \ell\right)\} \leq \mathbb{Q}^0(-y < S_t < f_t - w \ \forall t \in [\![0,\ell]\!]) =: Q(w,y,\ell)\,.$$

We deduce that, for any real $y \in [0, d_1 n]$, and any integer ℓ,

$$\sup_{0\leq z\leq y} \mathbb{Q}^z\left(\tau_f > \ell\right) \leq \sup_{j\in[\![0,M]\!]:x_j\leq y} Q(x_j,x_{j+1},\ell)\,.$$

Therefore, we have

$$T_2 \leq \max_{i\in[\![0,L-1]\!],\, j\in[\![0,M]\!]:\, x_j\leq f_{k_{i+1}}} \left\{ d_1\nu\left(1-\frac{i+1}{L}\right)^{1/3} + \limsup_{n\to\infty}\frac{1}{n^{1/3}} \ln Q(x_j,x_{j+1},n-k_i)\right\}.$$

By equation (4.2),

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \ln Q(x_j,x_{j+1},n-k_i) = -\frac{3\sigma^2\pi^2}{2d_1^2}\left(1-\frac{i}{L}\right)^{1/3} H(1/M,d_1,d_2)\,.$$

We deduce that

$$\begin{aligned} T_2 &\leq \max_{i\in[\![0,L-1]\!]} \left\{ d_1\nu\left(1-d_2\frac{i+1}{L}\right)^{1/3} - \frac{3\sigma^2\pi^2}{2d_1^2}\left(1-\frac{i}{L}\right)^{1/3} H(1/M,d_1,d_2)\right\} \\ &\leq \max_{t\in[0,1]} \left\{ d_1\nu\left(1-\frac{d_2}{L}-d_2 t\right)^{1/3} - \frac{3\sigma^2\pi^2}{2d_1^2}(1-t)^{1/3} H(1/M,d_1,d_2)\right\}. \end{aligned}$$

Let d_2 tend to 1, and L et M go to infinity to get

$$\begin{aligned} \liminf_{n\to\infty}\frac{1}{n^{1/3}}\ln P(Z_n>0) &\geq -\frac{3\pi^2}{2d_1^2} - \max_{t\in[0,1]}\left\{(d_1\nu - \frac{3\pi^2}{2d_1^2})(1-t)^{1/3}\right\} \\ &= -\frac{3\pi^2}{2d_1^2} - \max\left\{d_1\nu - \frac{3\pi^2}{2d_1^2}, 0\right\}. \end{aligned}$$

Taking $d_1 = \left(\frac{3\sigma^2\pi^2}{2\nu}\right)^{1/3}$ completes the proof of Theorem 1.2. $\square$

Acknowledgements : We are grateful to Zhan Shi for many useful discussions.

Bibliographie

[Aid08] E. Aidékon. Transient random walks in random environment on a Galton–Watson tree. *Probab. Theory Related Fields*, 142(3-4) :525–559, 2008.

[Ali99] S. Alili. Asymptotic behaviour for random walks in random environments. *J. Appl. Probab.*, 36(2) :334–349, 1999.

[AN72] K. B. Athreya and P. E. Ney. *Branching processes*. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.

[And06] P. Andreoletti. On the concentration of sinai's walk. *Stochastic Process. Appl.*, 116(10) :1377–1408, 2006.

[BAFGH08] G.B. Ben Arous, A. Fribergh, N. Gantert, and A. Hammond. Biased random walks on a Galton–Watson tree with leaves. *available at arXiv :0711.3686v3*, 2008.

[Big77] J. D. Biggins. Martingale convergence in the branching random walk. *J. Appl. Probability*, 14(1) :25–37, 1977.

[BK04] J. D. Biggins and A. E. Kyprianou. Measure change in multitype branching. *Adv. in Appl. Probab.*, 36(2) :544–581, 2004.

[BLSW91] J. D. Biggins, B. D. Lubachevsky, A. Shwartz, and A. Weiss. A branching random walk with a barrier. *Ann. Appl. Probab.*, 1(4) :573–581, 1991.

[CD87] D. Coppersmith and P. Diaconis. Random walks with reinforcement. *Unpublished manuscript*, 1987.

[CGZ00] F. Comets, N. Gantert, and O. Zeitouni. Quenched, annealed and functional large deviations for one-dimensional random walk in random environment. *Probab. Theory Related Fields*, 118(1) :65–114, 2000.

[Che67] A. A. Chernov. Reduplication of a multicomponent chain by the mechanism of "lightning". *Biophysica*, 12 :297–301, 1967.

[Che97] D. Chen. Average properties of random walks on Galton-Watson trees. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(3) :359–369, 1997.

[Col06] A. Collevecchio. Limit theorems for reinforced random walks on certain trees. *Probab. Theory Related Fields*, 136(1) :81–101, 2006.

[CR88] B. Chauvin and A. Rouault. Kpp equation and supercritical branching brownian motion in the subcritical speed-area. application to spatial trees. *Probab. Theory Related Fields*, 80(2) :299–314, 1988.

[CV06] F. Comets and V. Vargas. Majorizing multiplicative cascades for directed polymers in random media. *ALEA*, 2 :267–277, 2006.

[DGPS07] A. Dembo, N. Gantert, Y. Peres, and Z. Shi. Valleys and the maximum local time for random walk in random environment. *Probab. Theory Related Fields*, 137(3-4) :443–473, 2007.

[DGPZ02] A. Dembo, N. Gantert, Y. Peres, and O. Zeitouni. Large deviations for random walks on Galton-Watson trees : averaging and uncertainty. *Probab. Theory Related Fields*, 122(2) :241–288, 2002.

[DGZ04] A. Dembo, N. Gantert, and O. Zeitouni. Large deviations for random walk in random environment with holding times. *Ann. Probab.*, 32(1B) :996–1029, 2004.

[dH00] F. den Hollander. *Large deviations*, volume 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2000.

[Don89] R.A. Doney. On the asymptotic behaviour of first passage times for transient random walk. *Probab. Theory Related Fields*, 81 :239–246, 1989.

[DPZ96] A. Dembo, Y. Peres, and O. Zeitouni. Tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.*, 181(3) :667–683, 1996.

[DS84] P.G. Doyle and J.L. Snell. *Random walks and electric networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1984.

[DS07] B. Derrida and D. Simon. The survival probability of a branching random walk in presence of an absorbing wall. *Europhys. Lett. EPL*, 78(6) :Art. 60006, 6, 2007.

[ESZ07] N. Enriquez, C. Sabot, and O. Zindy. Limit laws for transient random walks in random environment on $\mathbb{Z}$. *available at arXiv :math/0703660v3*, 2007.

[Fel68] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. Wiley, New York, 2nd edition, 1968.

[Fel71] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 2. Wiley, New York, 2nd edition, 1971.

[Fra95] J. Franchi. Chaos multiplicatif : un traitement simple et complet de la fonction de partition. In *Séminaire de Probabilités, XXIX*, volume 1613 of *Lecture Notes in Math.*, pages 194–201. Springer, Berlin, 1995.

[GdH94] A. Greven and F. den Hollander. Large deviations for a random walk in random environment. *Ann. Probab.*, 22(3) :1381–1428, 1994.

[Gol84] A. O. Golosov. Localization of random walks in one-dimensional random environments. *Comm. Math. Phys.*, 92(4) :491–506, 1984.

[Gol86] A. O. Golosov. Limit distributions for a random walk in a critical one-dimensional random environment. *Uspekhi Mat. Nauk*, 41(2(248)) :189–190, 1986.

[Gro04] T. Gross. *Marche aléatoire en milieu aléatoire sur un arbre.* PhD thesis, 2004.

[GS02] N. Gantert and Z. Shi. Many visits to a single site by a transient random walk in random environment. *Stochastic Process. Appl.*, 99(2) :159–176, 2002.

[GZ98] N. Gantert and O. Zeitouni. Quenched sub-exponential tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.*, 194(1) :177–190, 1998.

[HH07] J. W. Harris and S. C. Harris. Survival probabilities for branching Brownian motion with absorption. *Electron. Comm. Probab.*, 12 :81–92 (electronic), 2007.

[HS] Y. Hu and Z. Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *To appear in Ann. Prob.*

[HS98] Y. Hu and Z. Shi. The limits of Sinai's simple random walk in random environment. *Ann. Probab.*, 26(4) :1477–1521, 1998.

[HS07a] Y. Hu and Z. Shi. Slow movement of random walk in random environment on a regular tree. *Ann. Probab.*, 35(5) :1978–1997, 2007.

[HS07b] Y. Hu and Z. Shi. A subdiffusive behaviour of recurrent random walk in random environment on a regular tree. *Probab. Theory Related Fields*, 138(3-4) :521–549, 2007.

[Igl74] D. L. Iglehart. Random walks with negative drift conditioned to stay positive. *J. Appl. Probability*, 11(4) :742–751, 1974.

[Kes78] H. Kesten. Branching Brownian motion with absorption. *Stochastic Processes Appl.*, 7(1) :9–47, 1978.

[Kes86] H. Kesten. The limit distribution of Sinaĭ's random walk in random environment. *Phys. A*, 138(1-2) :299–309, 1986.

[KKS75] H. Kesten, M. V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math.*, 30 :145–168, 1975.

[KLPP97] T. Kurtz, R. Lyons, R. Pemantle, and Y. Peres. A conceptual proof of the Kesten-Stigum theorem for multi-type branching processes. In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pages 181–185. Springer, New York, 1997.

[LDMF99] P. Le Doussal, C. Monthus, and D. Fisher. Random walkers in one-dimensional random environments : exact renormalization group analysis. *Phys. Rev.*, E 59 :4795–4840, 1999.

[Liu00] Q. Liu. On generalized multiplicative cascades. *Stochastic Process. Appl.*, 86(2) :263–286, 2000.

[LP92] R. Lyons and R. Pemantle. Random walk in a random environment and first-passage percolation on trees. *Ann. Probab.*, 20(1) :125–136, 1992.

[LP04] R. Lyons and Y Peres. *Probability on Trees and Networks.* Cambridge University Press, in progress. Current version published on the web at http ://php.indiana.edu/∼rdlyons, 2004.

[LPP95] R. Lyons, R. Pemantle, and Y. Peres. Ergodic theory on Galton-Watson trees : speed of random walk and dimension of harmonic measure. *Ergodic Theory Dynam. Systems*, 15(3) :593–619, 1995.

[LPP96] R. Lyons, R. Pemantle, and Y. Peres. Biased random walks on Galton-Watson trees. *Probab. Theory Related Fields*, 106(2) :249–264, 1996.

[Lyo92] R. Lyons. Random walks, capacity and percolation on trees. *Ann. Probab.*, 20 :2043–2088, 1992.

[Mog75] A.A. Mogul'skii. Small deviations in a space of trajectories. *Theory Probab. Appl.*, 19(4) :726–736, 1975.

[MP02] M. Menshikov and D. Petritis. On random walks in random environment on trees and their relationship with multiplicative chaos. In *Mathematics and computer science, II (Versailles, 2002)*, Trends Math., pages 415–422. Birkhäuser, Basel, 2002.

[Nev86] J. Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré*, 22(2) :199–207, 1986.

[Pem88] R. Pemantle. Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.*, 16(3) :1229–1241, 1988.

[Pet75] V. V. Petrov. *Sums of independent random variables.* Springer-Verlag, New York, 1975. Translated from the Russian by A. A. Brown, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82.

[Pet05] D. Petritis. Sequential and asynchronous processes driven by stochastic or quantum grammars and their application to genomics : a survey. *available at arXiv :math/0511346v1*, 2005.

[Pia98] D. Piau. Théorème central limite fonctionnel pour une marche au hasard en environment aléatoire. *Ann. Probab.*, 26 :1016–1040, 1998.

[PP95] R. Pemantle and Y. Peres. Critical random walk in random environment on trees. *Ann. Probab.*, 23(1) :105–140, 1995.

[PZ08] Y. Peres and O. Zeitouni. A central limit theorem for biased random walks on Galton-Watson trees. *Probab. Theory Related Fields*, 140(3-4) :595–629, 2008.

[SD08] D. Simon and B. Derrida. Quasi-stationary regime of a branching random walk in presence of an absorbing wall. *J. Stat. Phys.*, 131(2) :203–233, 2008.

[Shi98] Z. Shi. A local time curiosity in random environment. *Stochastic Process. Appl.*, 76(2) :231–250, 1998.

[Sin82] Ya. G. Sinaĭ. The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatnost. i Primenen.*, 27(2) :247–258, 1982.

[Sol75] F. Solomon. Random walks in a random environment. *Ann. Probab.*, 3 :1–31, 1975.

[Tem72] D. E. Temkin. One-dimensional random walks in a two-component chain. *Soviet math. Dokl.*, 13 :1172–1176, 1972.

[Vir00] B. Virág. On the speed of random walks on graphs. *Ann. Probab.*, 28(1) :379–394, 2000.

[VW08] V.A. Vatutin and V. Wachtel. Local probabilities for random walks conditioned to stay positive. *Probab. Theory Related Fields*, to appear, 2008.

[Zei04] O. Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.